Circles -- Halting Studies

1. Circles 0 and 1

A large fraction of my statistical studies have been done with N=1, and it is assumed here. It is also worth noting that statistical studies show that if a run is going to halt it usually does so within the first few placements.

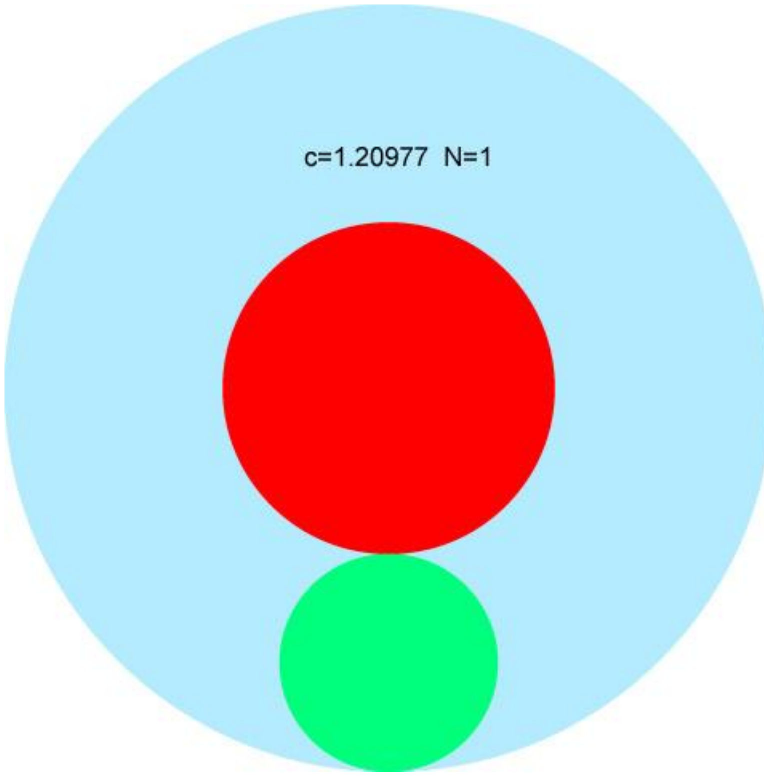

Fig. 1. A worst case for circles 0 and 1. The bounding region is blue. Circle 0 (red) is at the center, and circle 1 (green) is tangent to both circle 0 and the boundary.

Two kinds of worst cases are of interest: Ones where earlier circles maximally block later ones, and ones where earlier circles minimally block later ones. The construction of Fig. 1 is of the maximally-blocks kind. The red circle 0 will block all later circle 1 (green) locations if it is placed in the exact center and c > 1.20977. It is thus seen that *if c > 1.20977 the circle-in-circle algorithm is not unconditionally nonhalting* -- at least one halting configuration exists. If we look at Fig. 7.1 of "Fractalize That" [1] it is evident that c_1 cannot be higher than 1.20977. The value 1.20977… will be called c*.

This construction implies that as c increases beyond c* there is a halting probability which is > 0. In principle it can be calculated as a function of c.

Figure 7.2 of "Fractalize That" gives a c_1 value around 1.3, which differs substantially from 1.20977. This most likely arises from the limited resolution of the Monte Carlo method used for the data of Fig. 7.2. If the halting probability is very small (as it will be with c only slightly above 1.20977) the Monte Carlo method will not resolve it. Another limitation of the Monte Carlo method in low-probability situations is that practical computation requires that some upper limit be placed on the number of trials for a given placement. There is thus a question whether the probability is actually zero, or if in fact the expectation value of the number of trials for a placement is beyond the limit used in the code.

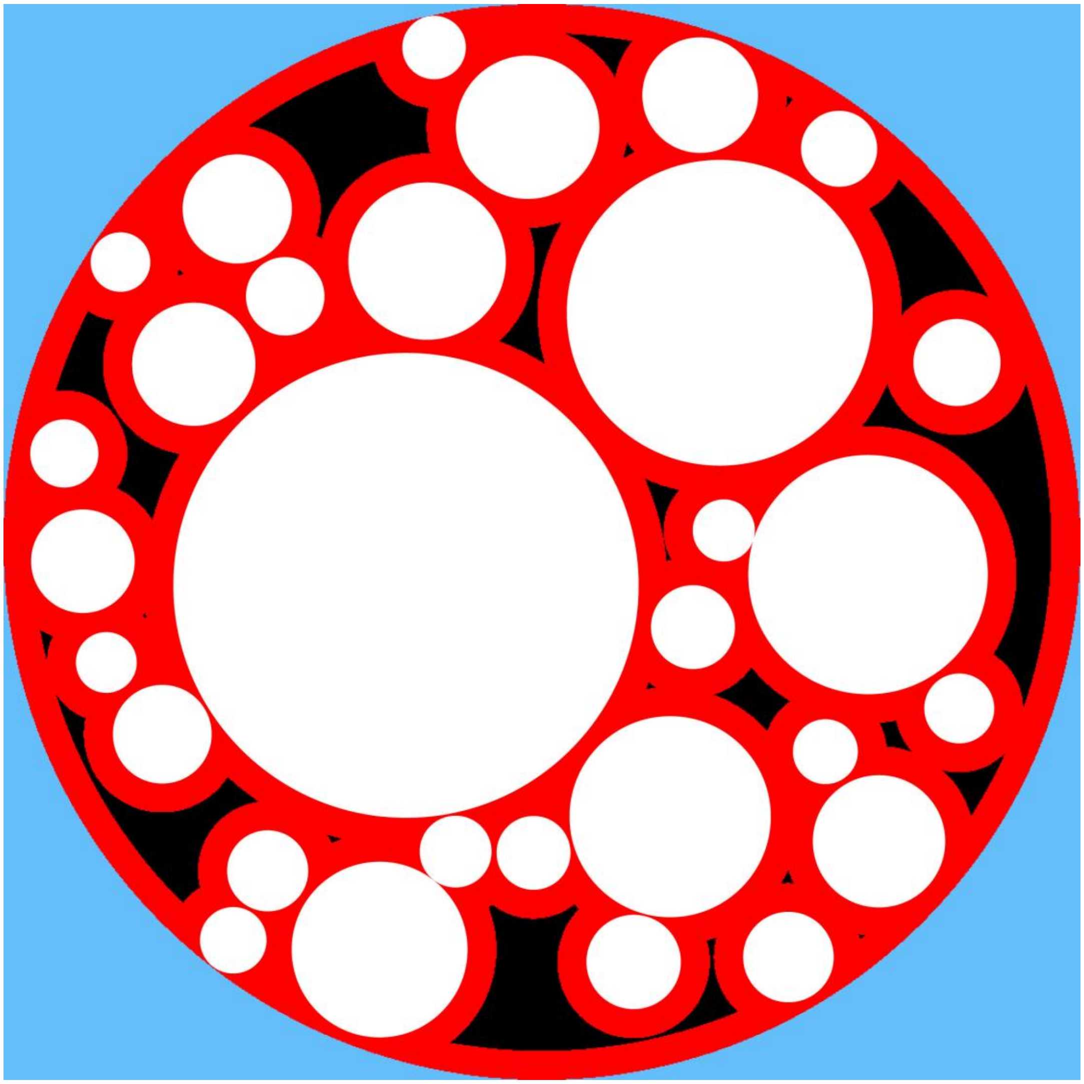

Fig. 2. An example of the algorithm run with the critical value c=1.20977 with N=1 (30 circles, circular boundary). The black region is available for the next placement. With this c value there appears to be plenty of room for further placements.

Figure 2 shows a typical run where red bands having a width equal to the next-to-be-placed radius surround each circle and the boundary. The black area is thus the region available for the next placement. Can it be proved that for c < 1.20977 there is always a place for the next placement for an arbitrarily large number of placements?

2. Circles 0, 1, 2

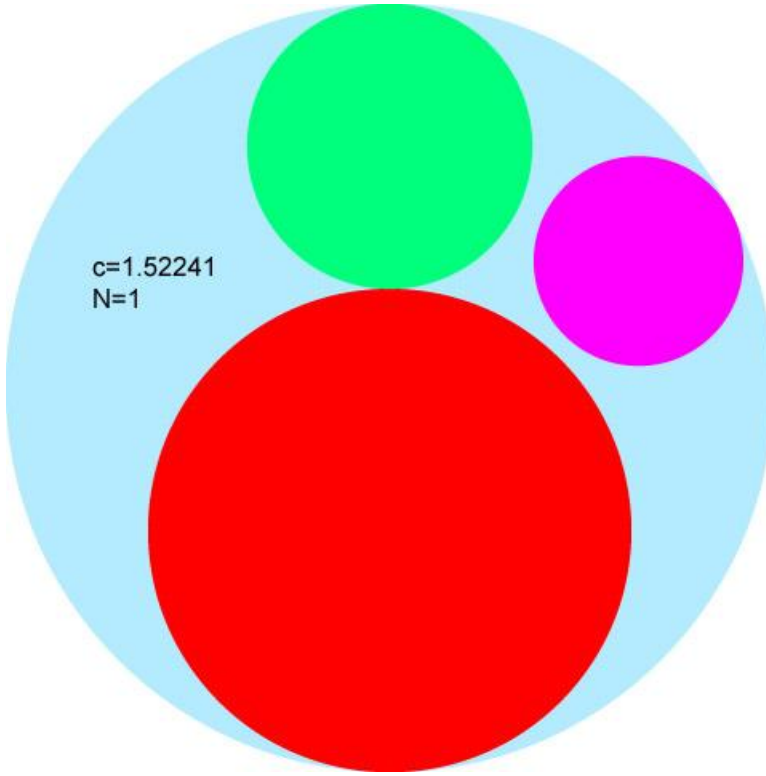

Fig. 3. A worst case for circles 0, 1, 2. The bounding region is blue. Circle 0 (red) is tangent to the boundary and circle 1, and circle 1 (green) is tangent to the boundary and circle 0. Circle 2 is magenta and is tangent to the boundary.

The main conclusion from Fig. 3 is that the algorithm is unconditionally halting when c > 1.52241 because of the placements of circles 0 and 1. Thus we can say (with reference to Fig. 7.1 of "Fractalize That") that c_2 cannot be higher than 1.52241. With this c value circles 0 and 1 can only have the positions shown, but many positions are possible for circle 2.

This construction implies that as c falls below 1.52241 there is a halting probability which is < 1 while for c > 1.52241 the algorithm always halts at placement 1.

These results will change substantially for other values of N. It is fairly straightforward to construct extreme examples with inclusive boundaries, and would be much harder with periodic boundaries. Figure 7.2 of "Fractalize That" [1] shows that the choice of periodic versus inclusive boundaries makes a very large difference.

3. Halting at Placement 1.

Figures 1 and 3 show extreme cases for placement of circles 0 and 1. Since most runs that halt do so early in the process it is interesting to trace out the halting probability versus c for circles 0 and 1. Code was constructed that looks only at this question. The x,y for circle 0 was chosen uniformly such that it does not overlap the circular boundary. Choices of x,y for circle 1 were made repeatedly until (a) a successful placement was made or (b) the (large) maximum allowed number of random trials was exceeded. Case (b) is interpreted as halting. For "high" c values where halting is common the calculations run very slowly because a quite large maximum number of allowed trials is needed and most of the time the search for circle 1 runs to the maximum number of allowed trials.

Each point in Fig. 4 represents 3000 random positions of circle 0, so that the statistics are reasonably good.

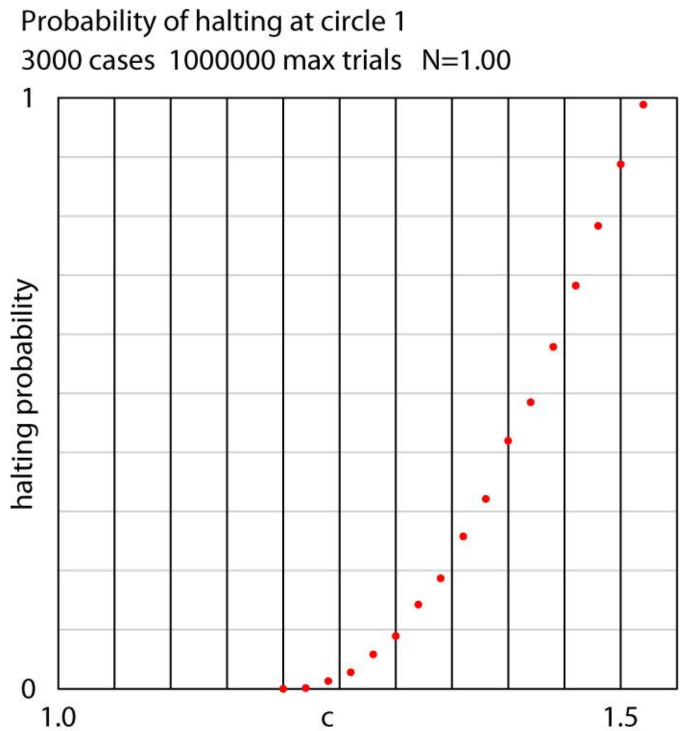

Fig. 4. A monte carlo calculation of the halting probability at placement 1 versus c.

The code does not assume the critical c values of 1.20977 or 1.52241, but the probability smoothly approaches the values 0 and 1 as the corresponding c values are approached. The results of the monte carlo calculations are thus in good agreement with the extreme examples in Figs. 1 and 3 (and with Fig. 7.2 of [1]). At this time I wonder if Fig. 3 isn't the whole story (at least when N=1) -- that failure occurs at placement 1 or not at all[1].

The curve defined by Fig. 4 appears to be asymmetric. The halting probability is 0 for c <1.20977 and is 1 for c > 1.52241 from the constructions of Figs. 1 and 3. The probability is approximately .5 when c = 1.42.

Efforts to prove that random circle fractals are unconditionally nonhalting should focus on the region c < 1.20977 (with N=1). Such a proof would require not only that there be no halting at placement 1 (as studied here), but that there also be none at placements 2, 3, … .

One can conclude from Fig. 3 that in determining the behavior of the halting probability curve (Fig. 3) near the upper limit at c = 1.52241 one can ignore circle 2 since it always fits. Thus an analytical approach near this limit would only need to consider circles 0 and 1.

4. Definition of the Dimensionless Gasket Area

For a statistical geometry fractal with bounding area A, the i-th shape area A_i (i = 0, 1, …) is given by

$$A_i = \frac{A}{\zeta(c,N)(i+N)^c} \qquad (1)$$

[1] The data for Fig. 7.2 of [1] was taken out to a substantial number of placements, with a rather high maximum number of trials. It has subsequently been understood that for good statistics the maximum trials must be *a substantial multiple of the normal average number of trials* for the given placement number. Since average trials rises steeply with placements, the max trials used there may not have been adequate for higher placement numbers.

We let A_{gask} be the gasket area after n placements and define the dimensionless gasket area $\Gamma(c,N,n)$ by

$$\Gamma(c,N,n)=\frac{A_{gask}}{nA_{n+1}}=\frac{A-\sum_{i=0}^{n}A_i}{nA_{n+1}}=\frac{A-\sum_{i=0}^{n}\frac{A}{\zeta(c,N)(i+N)^c}}{n\frac{A}{\zeta(c,N)(n+1+N)^c}}$$
$$=\frac{1-\sum_{i=0}^{n}\frac{1}{\zeta(c,N)(i+N)^c}}{\frac{n}{\zeta(c,N)(n+1+N)^c}}=\frac{\zeta(c,N)-\sum_{i=0}^{n}\frac{1}{(i+N)^c}}{\frac{n}{(n+1+N)^c}} \tag{2}$$

It is seen that Γ is a ratio of the gasket area after n placements to the product of the last-placed shape number n and the area of shape n+1. Γ is the mean area available for the next placement in units of the next-placed area A_{n+1}.

5. Numerical Study of the Dimensionless Gasket Area

Calculations of Γ have been made, using double-precision arithmetic. The tabulation below shows the results for N=2.

```
  values of gasket_area/(n_place*next_area)

      c=1.2000000 c=1.2500000 c=1.3333333
n=  1  21.937114   17.958609   13.995080
    2  13.413474   10.921224    8.434314
    4   9.179358    7.431837    5.686002
    8   7.078779    5.704498    4.330698
   16   6.035766    4.848462    3.661286
   32   5.516818    4.423121    3.329454
   64   5.258117    4.211258    3.164403
  128   5.128981    4.105550    3.082116
  256   5.064469    4.052754    3.041035
  512   5.032227    4.026372    3.020510
 1024   5.016109    4.013184    3.010251
 2048   5.008051    4.006591    3.005122
 4096   5.004022    4.003295    3.002556
 8192   5.002006    4.001647    3.001273
16384   5.000998    4.000823    3.000629
32768   5.000493    4.000411    3.000306
65536   5.000240    4.000204    3.000142
```

The first column is the placement number n, tabulated at points where n is a power of 2. Columns 2-4 use the c values listed at the head of each column. The reader can see that as n becomes large Γ appears to be converging toward the integers 5, 4, and 3 respectively. The following conjectures are offered based on computation:

$$\Gamma(c,N,n)>\frac{1}{c-1} \tag{3a}$$

$$Lim_{n\to\infty}\Gamma(c,N,n)=\frac{1}{c-1} \qquad (3b)$$

This is a quite simple relationship. Computations indicate that this relationship holds independent of n and N, i.e., it is only dependent on c. As n becomes large the > symbol can be replaced by ≅. It relates the amount of space available for placement directly to the power law exponent c. It is perhaps significant that this relationship does not depend on the shape, i.e., it holds for any shape. If area is replaced by volume, it also holds for the 3D case.

It is not at all obvious why complicated Eq. (2) should result in this very simple result. For the present one can explore the "never halts" question using Eq. (3) as appropriate. If this proves fruitful, a formal proof of Eq. (3) would be called for. Equation (3) can be rearranged to give:

$$\frac{A_{gask}}{n}>\frac{1}{c-1}A_{n+1} \qquad (4)$$

6. Is Halting Always at Circle 1?

Code was created to rapidly determine at what placement halting occurs for circles within a circle. Because of the observation that halting always occurs at an early stage, only the first 4 circles were placed. This greatly accelerates the computation. The criterion for halting was set at 1,000,000 unsuccessful trials.

With N=1 the results showed no halting at any placement other than circle 1. This was surprising. The code was checked with extreme cases like c=1.9 and N=6 and there was halting at placements beyond 1, indicating that the code is correct. A successful theory should account for this halting only at placement 1.

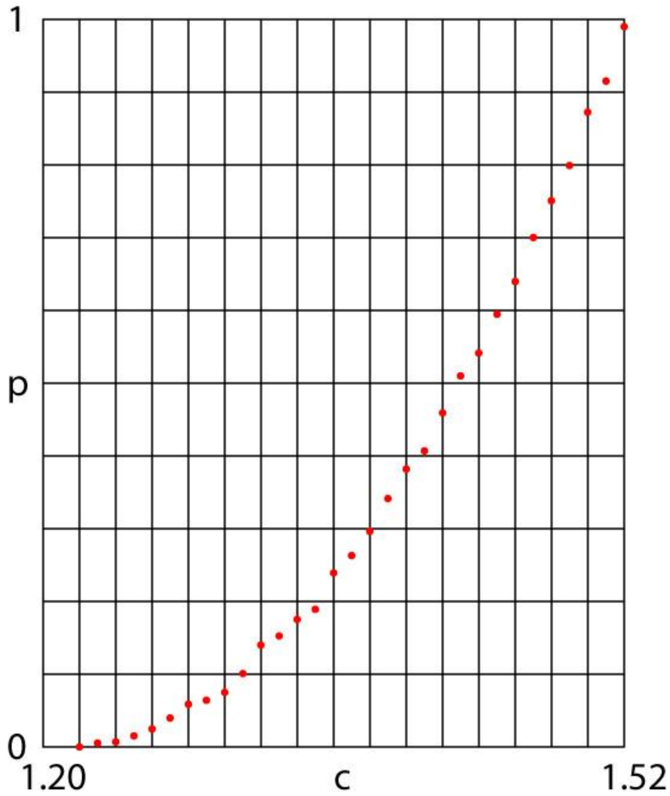

Fig. 5. A monte carlo calculation of the halting probability at placement 1 versus c with N=1. Each point represents 2000 runs.

7. The Placement Probability.

If the total "black" area (see Fig. 2) is A_b, and the area of the bounding circle less the halo width is A, we can define the placement probability p_{pl} as:

$$p_{pl} = \frac{A_b}{A} \qquad (5)$$

One can calculate the black area numerically after each placement, thus creating a record of the behavior of p_{pl} versus placement number n. Figure 6 shows an example.

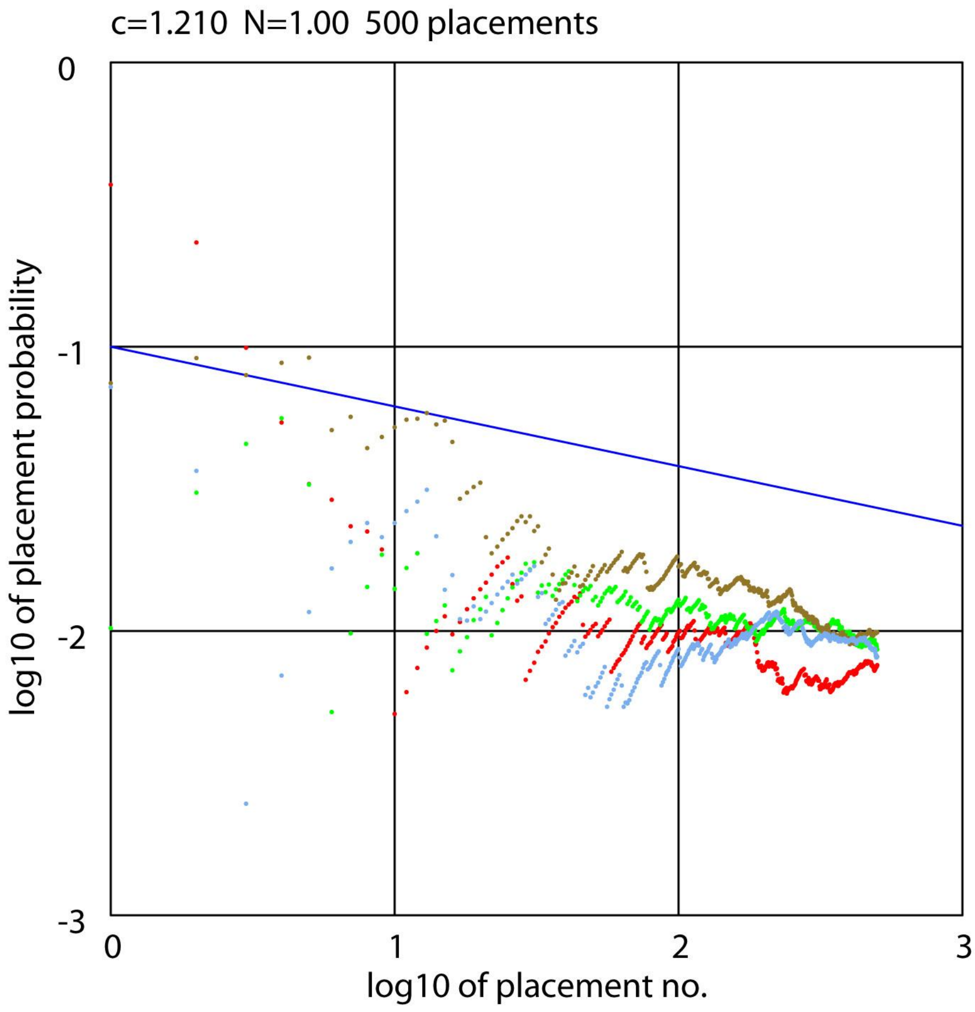

Fig. 6. The placement probability versus placed shapes n with c=1.21 and N=1, placements 1 to 500. There are 4 data sets for 4 runs, each with different-color dots. The blue line has a slope $-(c-1)$ which is the theoretical slope of the curve for large numbers of placements. The raster used for determining the black area was 256x256.

This is a noisy process, but finds simple qualitative explanations. It must be kept in mind that the red "halos" around each circle get narrower as the algorithm progresses. For those black areas which are unaffected by a new placement (most of them) this means that the associated black area increases. This can account for the episodes of steadily rising probability seen in Fig. 6. Occasionally the placement falls within a large black area, which sharply reduces the associated black area. This can account for the downward excursions of p_{pl}, which are often quite sharp and distinct. An especially large one occurs at placement 11.

It is interesting that the lowest probability occurs for n=3 for the blue data set, which supports the idea that "early" halting may be the dominant form of halting. By n=500 the different runs appear to be converging to a single behavior.

The amount of noise in a plot of this kind falls for large n because the process now has a large number of small, distinct black areas available, so that things "average out" and the large downward excursions seen for few placements are much smaller with many placements. The relative effect of changes in the halo width with placement number is smaller with many placements.

This is a stochastic process with memory. Events at a given placement affect later events. It is also evident that the placement probability at a given placement number has a statistical distribution of substantial width.

If the cumulative trials versus placement number is a power law with exponent $+c$ (as is usually observed for circles), the relationship in Fig. 6 should be a power law with exponent $-(c-1)$. In a log-log plot a power law with this exponent has the slope of the blue line in Fig. 6. About all one can say is that the data is "not incompatible" which such a power law, given the noise and scatter. A comparable data set for the 1D case can be seen in Fig. 8.8 of "Fractalize That".

8. References.

[1] "Fractalize That", John Shier (2013-2014, various printings), privately printed book.

The best published source for a mathematical description of the algorithm is:

"An Algorithm for Random Fractal Filling of Space", John Shier and Paul Bourke
Computer Graphics Forum, Vol. 32, Issue 8, pp. 89-97, December 2013.

Copies of the last version of the paper to go to the editor can be downloaded from the author's web site (or that of Paul Bourke). The most recent publication on statistical geometry fractals is

Dunham and Shier, "The Art of Random Fractals" in Proceedings of the 2014 Bridges conference, Seoul, Korea (August 2014).

It can be viewed at the Bridges web site.