A New Recursion for Space-Filling Geometric Fractals

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Abstract. A recursive two-dimensional geometric fractal construction based upon area and perimeter is described. For circles the radius of the next circle is a constant γ times (gasket[1] area)/(gasket perimeter). Placement of the circles within a bounding circle is done by nonoverlapping random search. The nth circle area A_n goes over to a power law $A_n \sim n^{-c}$ for large *n*, where *n* is the number of placed circles and *c* is a constant. For circles it is found that if γ is a quotient of integers, *c* is also a quotient of integers. Computational evidence indicates that this algorithm is space-filling, and that it is nonhalting over a substantial range of γ values. A study of the three-dimensional case of spheres shows similar behavior. Limited studies of shapes other than circles show that the algorithm works for a variety of shapes.

1. Introduction

In previous work [2] a space-filling geometric algorithm was described, based on (a) a sequence of shapes whose areas obey a negative-exponent power law versus shape number *n*, and (b) placement of successive ever-smaller shapes by nonoverlapping random search.

Quite early in the study of these fractals it was concluded that the quantity

$$w = \frac{gasket\ area}{gasket\ perimeter} \qquad (1)$$

can be viewed as a measure[2] of the average gasket width. For circles one can define the *dimensionless* average gasket width as

$$b(c,N,n) = \frac{gasket\ area\ after\ n\ placements}{(gasket\ perimeter\ after\ n\ placements)(circle\ n+1\, diameter)} \qquad (2)$$

Numerical studies for the statistical geometry algorithm [2] showed that *b* is nearly independent of *n* for large *n*. This was viewed as evidence that the available space in the gasket falls in direct proportion to the circle diameter, so that the algorithm does not halt. The present study builds upon this idea. If *w* is a valid measure of average gasket width, then it should be possible to fill a spatial region with a sequence of ever-smaller circles according to a *recursive* rule for the circle radius r_{n+1}:

$$r_{n+1} = \gamma \left[\frac{gasket\ area\ after\ n\ placements}{gasket\ perimeter\ after\ n\ placements} \right] \qquad (3)$$

where γ is a dimensionless parameter. The gasket area is always positive and is a diminishing function of *n*, while the gasket perimeter grows without limit as *n* increases. Thus the circle radius r_n rapidly decreases.

2. Mathematical Properties of the Recursion.

The reader should keep in mind that the conclusions presented here are based on *computational experiments*. Proofs (or refutations) of the claims will require further study.

We first consider the behavior of the recursion of Eq. (3) before considering whether nonoverlapping random search can fill a region with circles having radii generated this way. Recursive *r* values were computed

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[1] The gasket is the part of the bounding region not within a placed circle, e.g. the white region between circles in Fig. 3(a). Study of the Shier-Bourke paper [2] will make it much easier to understand the results described here.

[2] The ratio *w* as a measure of gasket width is a difficult idea for many people. Much mental sumo wrestling may be needed before it is comfortable. Its usefulness is supported by the results reported here and in [2].

for circles using double-precision arithmetic, with the results shown in Fig. 1. It can be seen that in log coordinates both of the plotted quantities go very rapidly to a straight line as n increases, indicating that the data follows a power law whose exponent is the slope[3] of the line.

With $\gamma = 2.5$ (Fig. 1) the circle area data (red) has a slope –1.384613 for large n (with accuracy of 6 significant digits). The gasket area (blue) has a slope of –.384613. Such exponents differing by 1 have been cited previously [3-5] in studies of random Apollonian fractals. We let c be the (positive) exponent[4] for circle area, which is thus $c = 1.384613$. It is found that 1.384613 = 18/13 (to 6 significant digits) when $\gamma = 2.5$. The large-n slope of the data in Fig. 1 is found to be independent of the details of the first few circle radii, so that the first few recursions can have different rules to aid in launching (see sec. 4). The sizes of the first few circles have a significant effect on the ability of the algorithm to run at high γ values.

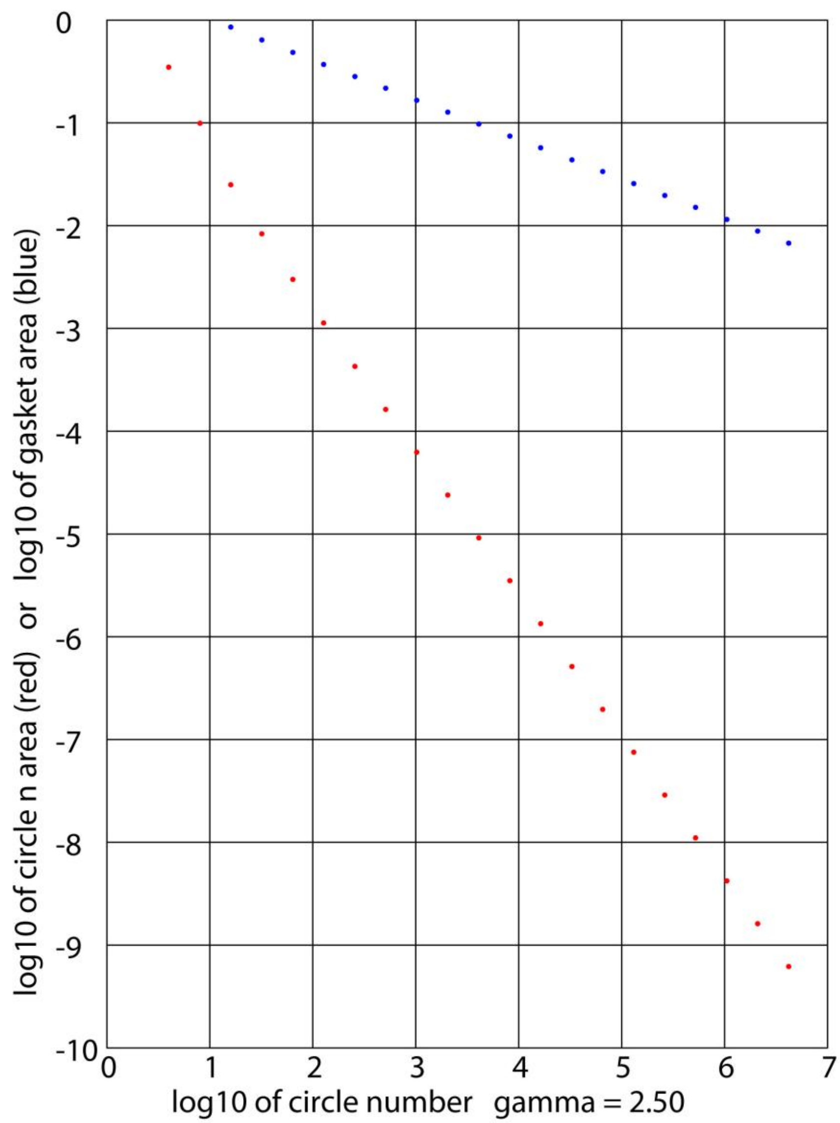


Fig. 1. Log-log plot of n^{th} circle area (red) and the gasket area (blue) versus the circle placement number n with $\gamma = 2.5$.

In general if a quotient of integers is chosen for γ, c is also a quotient of integers, with the c value given by

[3] If $y = x^c$, then $\log(y) = c \log(x)$, so that a plot of $\log(x)$ versus $\log(y)$ is a straight line with slope c.
[4] Thus the symbol c follows the usage for statistical geometry fractals [2].

$$c = \frac{4+2\gamma}{4+\gamma} \qquad \text{(4a)}$$

$$\gamma = -4\frac{c-1}{c-2} \qquad \text{(4b)}$$

Equation (4a) was found by trial-and-error and implies that $1 < c < 2$. Thus the fractal dimension $D = 2/c$ must obey $D > 1$. Equation (4a) agrees with computed c values to 6 significant decimal places. It should be possible to derive Eq. (4a) from Eq. (3) in the limit of large n.

The power law in n obeyed by the gasket area (blue points in Fig. 1) indicates that the gasket area can be made smaller than any given value if n is sufficiently large. This is the basis for the claim that *the algorithm is space-filling* if the placed circles do not overlap.

3. Circle Placement by Nonoverlapping Random Search.

Nonoverlapping random search is used in the statistical geometry algorithm [2]. Figure 2 shows the flow chart for this algorithm.

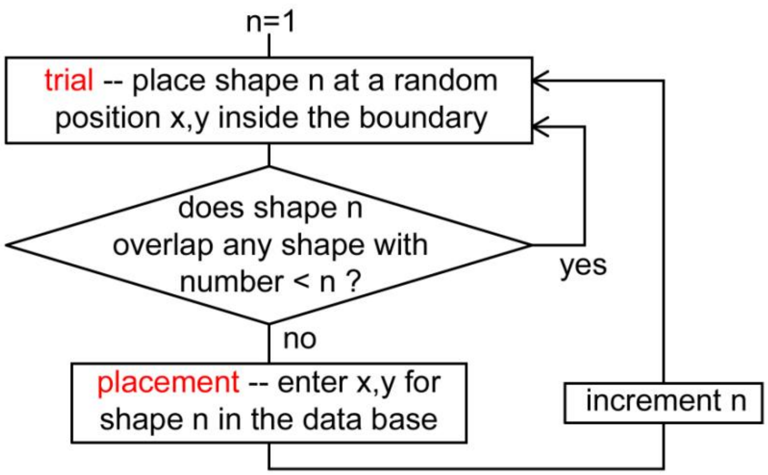


Fig. 2. Flow diagram for nonoverlapping random search. The shapes (here circles) get smaller as n is incremented following Eq. (3). The diagram has no STOP because available evidence indicates that it does not halt for a sizable range of γ values. Successive shapes (here circles or spheres) have radii calculated using Eq. (3) or Eqs. (5-7) or Eqs. (8-9).

Can nonoverlapping random search proceed without halting with the recursive rule of Eq. (3)? Figure 3 shows an example.

The algorithm does sometimes halt, subject to a halting probability (like statistical geometry [2]). A study of the halting probability for circles in a circle was made such that nonhalting runs stop at 50 circles ($\lambda = .4$ and $n_{tran} = 6$ as defined in sec. 4.). A run was viewed as halting if 6,000,000 trials were made without a placement. The results were as follows:

With $\gamma = 1.60$ 0 of 100 runs halted
With $\gamma = 1.80$ 0 of 100 runs halted
With $\gamma = 2.00$ 2 of 100 runs halted at placements 6 6
With $\gamma = 2.20$ 18 of 100 runs halted at placements 6 7 7 7 7 6 7 6 6 4 4 6 8 6 7 8 3 4
With $\gamma = 2.40$ 45 of 100 runs halted at placements 6 3 6 8 6 6 6 3 6 7 5 6 6 5 6 8 8 5
6 4 11 6 3 7 6 6 7 7 6 4 6 6 5 6 5 4 6 6 5 6 6 4 6 7 5

It is concluded that for low γ values the halting probability is close to zero. For higher γ values some fraction of the runs halt, and when they halt they do so at a small number of placements. This pattern of behavior is similar to that for statistical geometry [2].

The use of nonoverlapping random search with Eq. (3) requires special consideration for the first circle size. What should the initial area and perimeter be? It is reasonable (but somewhat arbitrary) to take the starting area and perimeter to be their values for the bounding circle. With the assumption of constant γ we must have $\gamma < 2$ for the first placed circle since otherwise the first circle would be larger than the bounding circle. With constant γ in most cases the first circle occupies a dominant part of the bounding area. More interesting results are seen if (only) the first few circles depart from Eq. (3) as in Eqs. (5-7). Figure 3(a) shows an example where $\gamma = 0.5$ for the first circle and $\gamma = 5/3$ otherwise.

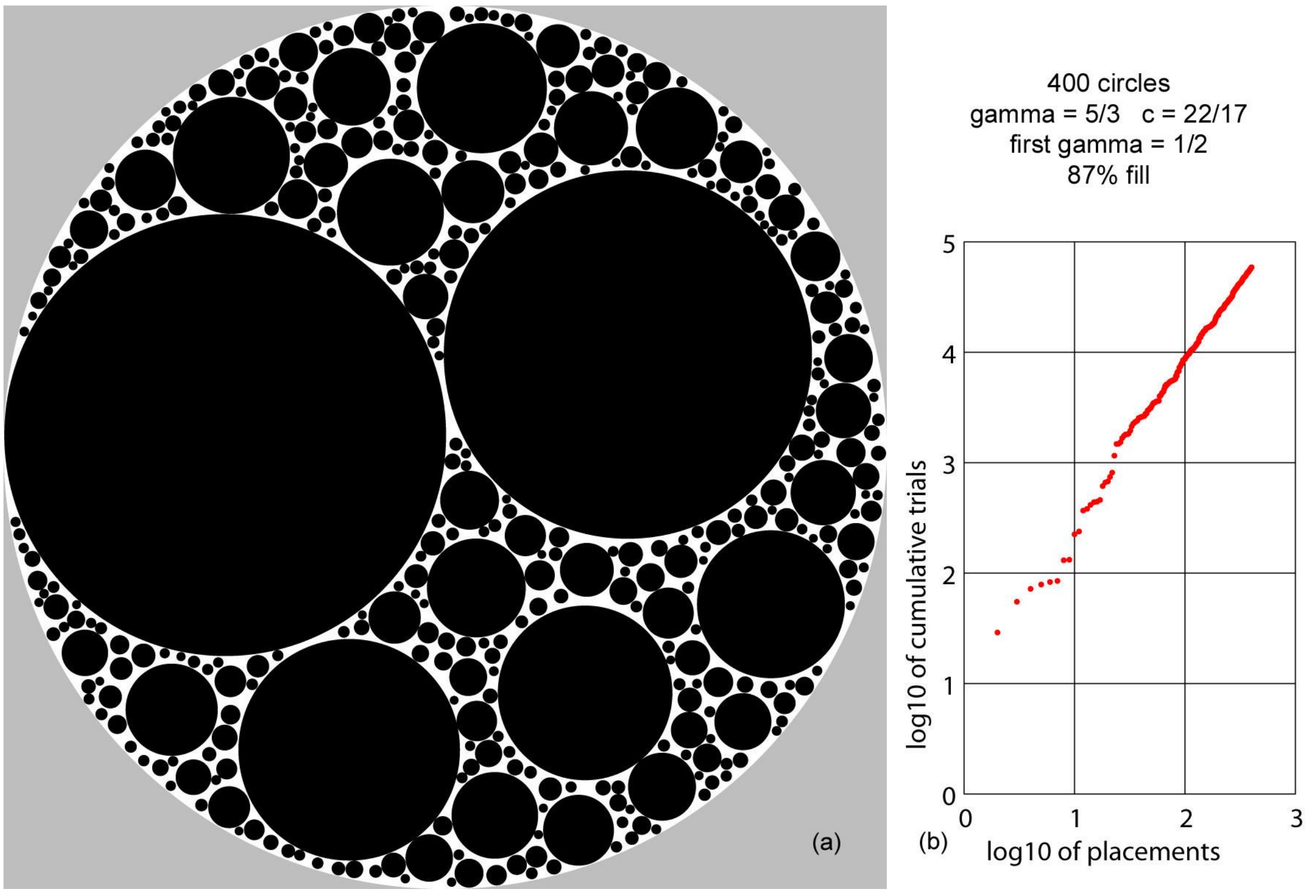


Fig. 3. (a) A space-filling pattern of 400 circles in a circle, $\gamma = 5/3$, $c = 22/17$, 87% fill. The first γ value was 1/2. (b) A log-log plot of the cumulative number of random trials needed to place n circles.

Figure 3(a) looks much like patterns made with the statistical geometry algorithm. In Fig. 3(b) the data approaches a straight line for large n, indicating that the cumulative number of trials versus n can be satisfactorily approximated by a positive-exponent power law. Such a power law never becomes infinite; thus the conclusion that *the algorithm does not halt* in this example. A formal proof of nonhalting for a range of γ values would be of much interest.

4. Launching.

It is found that one can define the first few circles in a variety of ways while the algorithm continues to run and converge to a power law (with the same exponent) for circle area after a large number of placements. The treatment of the first few circles will be called "launching" (some might call it "seeding").

Alternative radius rules are defined here. In Eqs. (5-7) we have parameters λ and n_{tran} in addition to γ. The parameters λ and n_{tran} only affect the first few circles; after than Eq. (3) holds for all n.

When $n = 1$: $r_1 = \lambda \dfrac{(area\ of\ boundary)}{(perimeter\ of\ boundary)}$ (5)

When $1 < n < n_{tran}$: $r_{n+1} = \left[\lambda + \left(1 - \left(\dfrac{n_{tran} - n}{n_{tran} - 1} \right)^2 \right) (\gamma - \lambda) \right] \left[\dfrac{gasket\ area\ after\ n\ placements}{gasket\ perimeter\ after\ n\ placements} \right]$ (6)

When $n \ge n_{tran}$: $r_{n+1} = \gamma \left[\dfrac{gasket\ area\ after\ n\ placements}{gasket\ perimeter\ after\ n\ placements} \right]$ (7) (same as (3))

Equation (6) causes the "effective" γ value to increase smoothly (as a parabola) from λ to γ over several steps. This flexibility in launching allows relatively high γ values to be used without early halting. Figure 4 shows an example of the high γ values possible with the scheme of Eqs. (5-7).

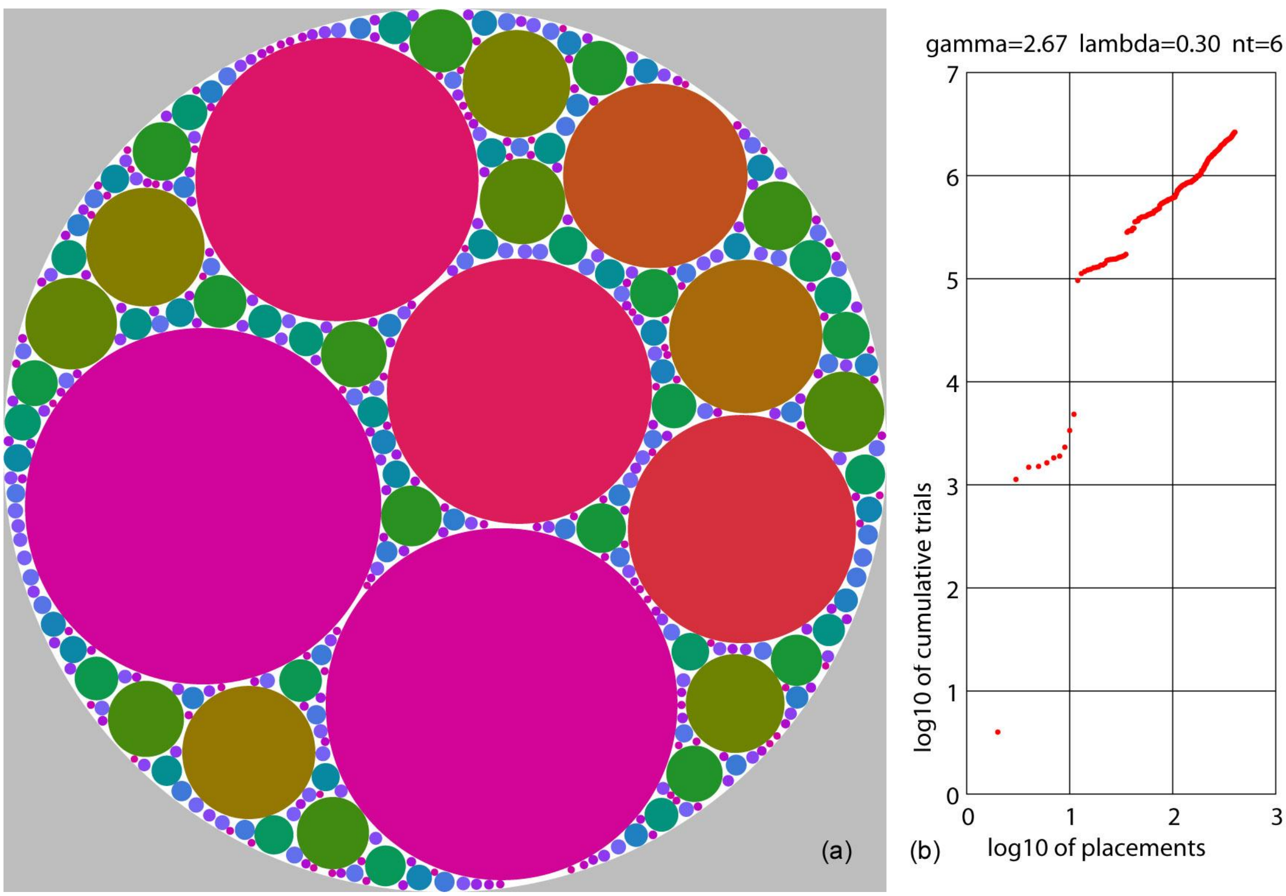


Fig. 4. (a) A space-filling pattern of 400 circles in a circle, $\gamma = 8/3$, $c = 7/5$, $\lambda = 0.3$, $n_{tran} = 6$. 94% fill. (b) A log-log plot of the cumulative number of random trials needed to place n circles. Log-periodic color with one full cycle.

For statistical geometry a cumulative trials plot such as Fig. 4(b) is a straight line with noise. Here we have a "stair step" behavior in which many trials are needed to place a few of the circles. For $n > 100$ the data in Fig. 4(b) can be approximated by a straight line, which is evidence for nonhalting.

Another launching scheme calls for computing two next-radius values:

$r_{n+1}^{A} = \delta R$ (8)

$$r_{n+1}^{B} = \gamma \left[\frac{gasket\ area\ after\ n\ placements}{gasket\ perimeter\ after\ n\ placements} \right] \tag{9}$$

and choosing the smaller of these values. Here δ is a dimensionless parameter and R is the radius of the bounding circle. As the process begins r^B is quite large, but it decreases steadily, eventually becoming smaller than the fixed value r^A.

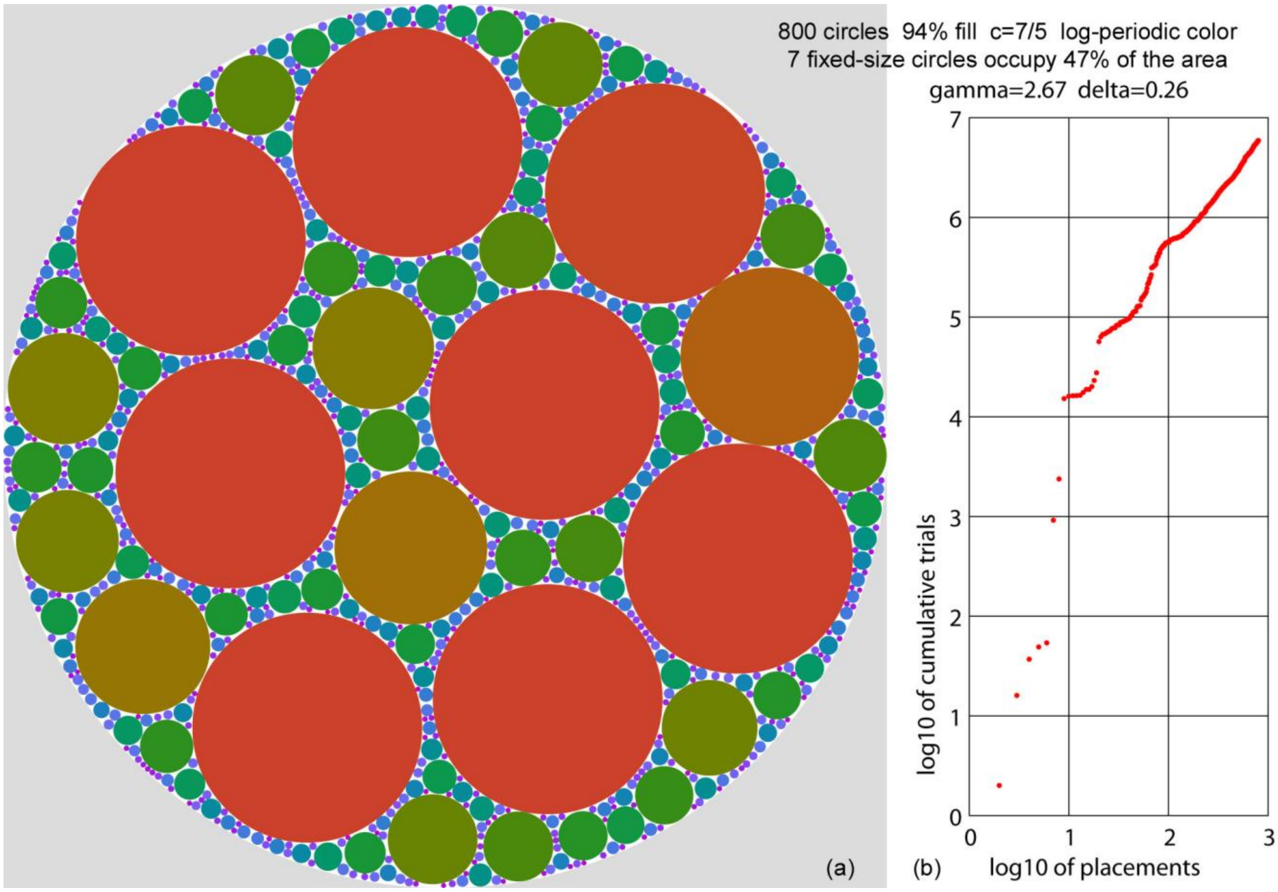


Fig. 5. (a) A space-filling pattern of 800 circles in a circle with the launching scheme of Eqs. (8) and (9). $\gamma = 8/3$, $\delta = .26$. 94% fill. (b) A log-log plot of the cumulative number of random trials needed to place *n* circles. Log-periodic color with one full cycle. The seven largest circles have the same radius.

Figure 5 shows a high-γ pattern with the launch scheme of Eqs. (8) and (9). The area occupied by the fixed-radius circles is typically around 50% of the boundary area. Figure 5(b) shows the same kind of stair-step behavior seen for other examples.

5. Three Dimensions: Spheres.

The obvious generalization of Eq. (3) to spheres in three dimensions is

$$r_{n+1} = \gamma \left[\frac{gasket\ volume\ after\ n\ placements}{gasket\ surface\ area\ after\ n\ placements} \right] \tag{10}$$

Figure 5 shows a typical example of iterating Eq. (10) for spheres (with the launch scheme of Eqs. (5-7)). The value γ = 2.25 is about as high as one can go for spheres. For large *n* the slope of the sphere volume data is

$c = -1.05555$ (= 19/18) while the slope of the gasket volume is $-.05555$. As for circles, if γ is a quotient of integers, c is also a quotient of integers (accurate to 6 significant digits). Data for c versus γ is given in Table 1.

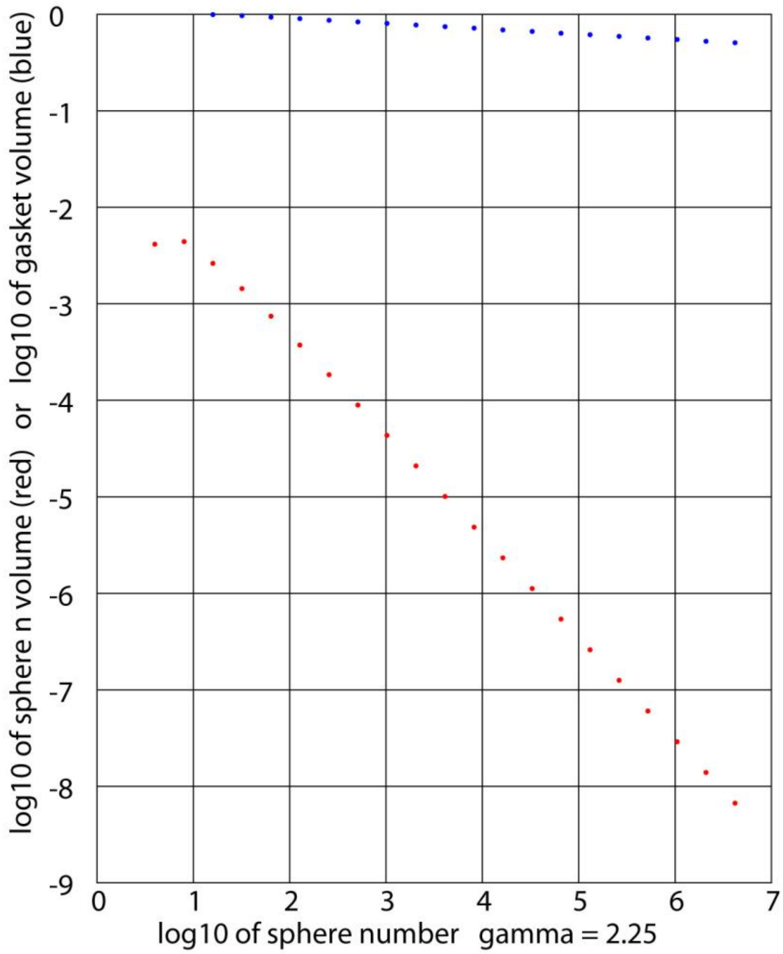


Fig. 6. Log-log plot of the volume of sphere n (red) and of the gasket volume (blue) versus the sphere number n.

The launching sequence for the spheres in Fig. 6 follows Eqs. (5-7) with the obvious adjustments for spheres. The negative-exponent power law seen for the gasket volume in Fig. 6 indicates that a nonoverlapping fractal based on Eq. (10) will be space-filling, but it will fill *very* slowly.

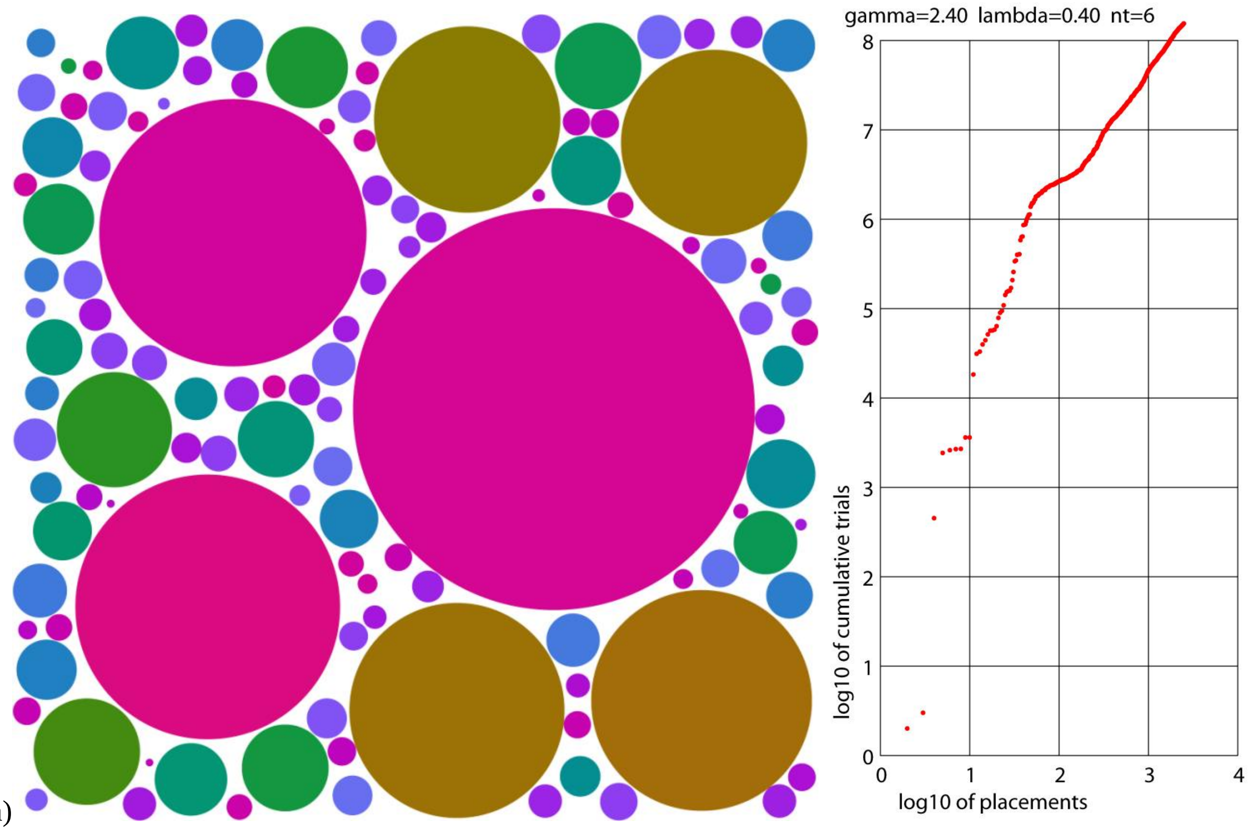
Table 1. Power-law slope c and fractal dimension D for sphere volume versus γ. The fractal dimension is $3/c$ [2].

γ	c	fractal D
1.5 = 3/2	27/26	78/27 = 2.888
1.75 = 7/4	165/158	474/165 = 2.872
2.0 = 2	21/20	60/21 = 2.857
2.25 = 9/4	19/18	54/19 = 2.842
2.5 = 5/2	87/82	246/87 = 2.827

The data of Table 1 has been found by trial and error to agree with the equation

$$c = \frac{36+3\gamma}{36+2\gamma} \qquad (11)$$

It is seen from Eq. (11) that $1 < c < 3/2$. The fractal dimension D must be > 2. It is conjectured that the coefficients in Eqs. (4) and (11) are integers, and that the 3 and 2 in Eq. (11) are Euclidean dimensions.



(a) (b)

Fig. 7. (a) Cross section of a space-filling pattern of 2500 spheres in a cube with $\gamma = 2.4$, $\lambda = 0.4$, $n_{tran} = 6$, $c = 18/17 = 1.0588$, 75% volume fill. The section runs through the cube center in the *xy* plane. Log-periodic color with one full cycle, with the largest and smallest spheres having the same color. The smallest shapes are spheres which have only a small intersection with the plane of the cross section (b) A log-log plot of the cumulative number of random trials needed to place *n* spheres for the same computer run.

The recursion (10) can produce a space-filling fractalization of spheres for γ values up to about 2.5. The trend in Fig. 6(b) is evidence that the algorithm does not halt. A formal proof of nonhalting is an interesting challenge.

Figure 8 shows another sphere-in-cube example, this time using the launch scheme of Eqs. (8) and (9). With these parameters the halting probability was high and this is a "lucky" survivor run. The first 25 spheres all have the same radius and have a magenta color. The runs which halted all did so at a placement fairly close to 25. Since this is a cross section, some magenta circles are small, which reflects the fact that the plane of the cross section cuts through only a small part of the corresponding sphere. The curve of Fig. 8(b) divides into two regimes. For low *n* values the number of trials needed to place the fixed-size spheres rapidly increases. After the area/perimeter rule takes over the curve flattens and we see a fairly even increase of trials in log-log coordinates. The trend in Fig. 8(b) shows that for large *n* the number of trials needed to achieve a given number of placements is quite well-behaved and is accurately described by a power law. Such a power law is evidence for nonhalting. The exponent of this power law was found to be 1.166 based on least-square fitting of the last 15000 points in the log-log data.

The launch sequences which have been explored here hardly exhaust the possibilities. Is there a launch sequence which is "optimum" or "best" in some sense? Can one achieve interesting results by pre-placement of the first few shapes?

The use of periodic boundaries resulting in patterns which can be tiled (see [2]) is one of the interesting studies which remain to be done.

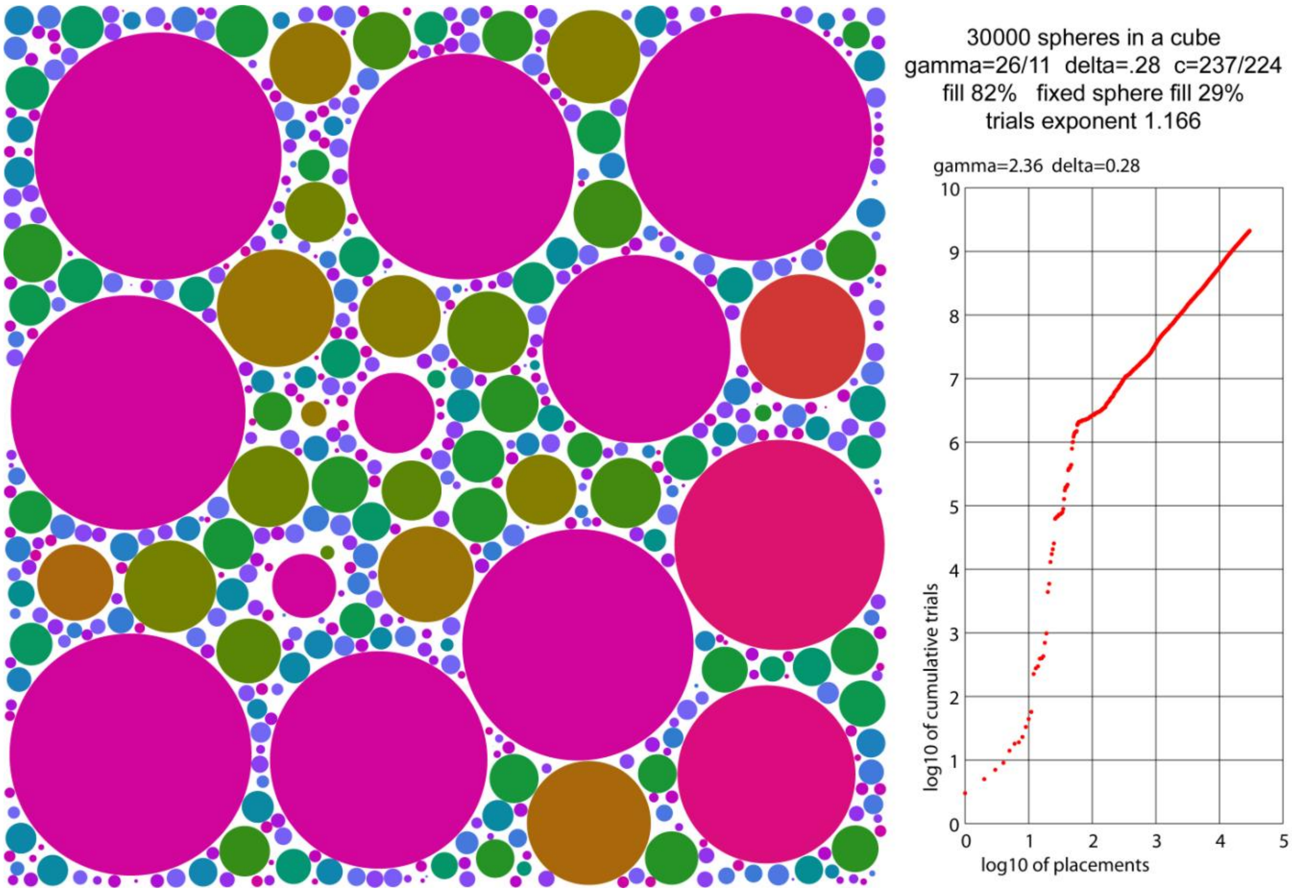


Fig. 8. (a) Cross section of a space-filling pattern of 30000 spheres in a cube with $\gamma = 26/11$, $\delta = .28$, $c = 237/224 = 1.058036$, 82% volume fill. The section runs through the cube center in the *xy* plane. Log-periodic color with one full cycle. (b) A log-log plot of the cumulative number of random trials needed to place *n* spheres for the same computer run.

6. Discussion.

One of the interesting open questions about the statistical geometry algorithm [2] is whether the nonoverlapping random search method can be used for a space-filling[5] construction with a different area rule. Such a different rule is presented here. A power-law rule for the areas of the circles is not assumed *a priori*, but it is found that such a power law describes the data quite accurately in the limit of a large number of shapes.

The algorithm is quite simple. Once launched, it requires only the ideas of perimeter, area, and volume, plus a single dimensionless parameter γ. It reveals an interesting property of area and perimeter (or volume and area). It is not obvious how one would use this idea in one dimension.

Many recursive fractals are known (e.g., Sierpinski) in which each recursion creates *k* times more (smaller) shapes than the previous recursion, i.e., the recursion is between "generations". This recursion is not generation-to-generation, and each shape has a different area.

The algorithm has some unique properties. It can create a space-filling power-law fractal based solely on perimeter-area-volume. The definition and calculation of power laws with exponents other than integers or half-integers is ordinarily done with logarithms. Iteration of Eq. (3) can generate such a negative-exponent power law without reference to logarithms.

The observation that if γ is a quotient of integers, *c* is also a quotient of integers is seen in Eq. (4a) and Eq. (11). The coefficients in Eqs. (4a) and (11) are conjectured to be integers. Derivation of Eq. (4a) from Eq. (3) would be interesting and should be possible. Formally this requires demonstration of the limit:

[5] I have avoided use of the word "packing" and used "filling". Packing usually refers to geometric constructions where the elements touch each other, which is not the case here.

$$\frac{\log\left(\pi r_{n+1}^2\right)-\log\left(\pi r_n^2\right)}{\log(n+1)-\log(n)} \to \frac{4+2\gamma}{4+\gamma} \quad as\ n\to\infty \qquad (12)$$

While circles and spheres are studied here, limited work with other shapes has been successful. It is conjectured that the algorithm runs without halting for any placed shape[6] over some range $0 < \gamma < \gamma_{max}$.

One description for this work and that of [2] is "non-Apollonian geometry".

7. An Almost-Identity.

Why should the computed exponents in Fig. 1 differ by exactly 1? Wouldn't it be interesting if one could find a β value such that

$$\frac{1}{\zeta(c)}\sum_{i=1}^{n}\frac{1}{i^c}+\frac{1}{n^\beta}\equiv 1 \qquad (13)$$

where $\zeta(\ )$ is the Riemann zeta function. One can come quite close by taking $\beta = c - 1$. This relationship for β versus c is used in certain published papers (e.g., [3-5]). If you compute the left-hand side of Eq. (13) you find

n	1st term	2nd term	sum
100	0.787372	0.251189	1.038561
1000	0.893290	0.125893	1.019182
10000	0.946511	0.063096	1.009607
100000	0.973192	0.031623	1.004814
1000000	0.986564	0.015849	1.002413

The sum differs from 1 by only a small amount, and it appears (without proof) that if you take n big enough the difference from 1 goes asymptotically to zero. The relation $\beta = c - 1$ is valid for the data of Fig. 1 and Fig. 6 if β is the exponent for the gasket area.

8. References

[1] J. Shier, Hyperseeing (Proceedings of the 2011 ISAMA Conference), June, p. 131 (2011).

[2] J. Shier and P. Bourke, Computer Graphics Forum, **32**, Issue 8, 89 (2013). This is the first publication in a refereed journal. The last version sent to the editor can be downloaded at [6] or [7].

[3] P. Dodds and J. Weitz, Phys. Rev. E, **65**, 056108-1 (2002)

[4] P. Dodds and J. Weitz, Phys. Rev. E, **67**, 016117-1 (2003)

[5] G. Delaney, S. Hutzler, and T. Aste, Phys. Rev. Letters, **101**, 120602 (2008).

[6] P. Bourke, web site http://paulbourke.net/fractals/

[7] J. Shier, web site http://john-art.com/stat_geom_linkpage.html

[6] Equation (3) can be formulated for any shape, but requires that a somewhat arbitrary length must be specified on the left-hand side of the equation. For example, for equilateral triangles one could choose the edge length or the center-to-vertex length. The choice of this length affects the scale of γ values (see Appendix A).

Appendix A -- Scaling

It is possible to put Eq. (3) into a more universal form. Let L be a length parameter such as the edge length of a square. We then define the area for the shape to be μL^2 and the perimeter to be σL, where μ and σ are dimensionless parameters ($\mu = \pi$ for a circle if L is the radius). If we have a square and take L to be the edge length we find $\mu = 1$ and $\sigma = 4$. If we instead take the length L to be the half-edge length, then $\mu = 4$ and $\sigma = 8$. The combination μ/σ^2 will be the same regardless of the choice made for the length parameter. Using μ and σ Eq. (3) becomes

$$L_{n+1} = \gamma \frac{\mu L_0^2 - \mu \sum_{i=1}^{n} L_i^2}{\sigma L_0 + \sigma \sum_{i=1}^{n} L_i} = \gamma \frac{\mu}{\sigma} \frac{L_0^2 - \sum_{i=1}^{n} L_i^2}{L_0 + \sum_{i=1}^{n} L_i} \qquad \text{(A1)}$$

where L_0 is the length parameter for the boundary shape. If we define a new parameter g by

$$g = \gamma\mu / \sigma \quad i.e. \quad \gamma = g\sigma / \mu \qquad \text{(A2)}$$

then

$$L_{n+1} = g \frac{L_0^2 - \sum_{i=1}^{n} L_i^2}{L_0 + \sum_{i=1}^{n} L_i} \qquad \text{(A3)}$$

When expressed in terms of L and g one can calculate universal results, and get the corresponding γ values for a particular case using Eq. (A2).

The combination μ/σ^2 changes if we consider a different shape. For example if we have a rectangle with length 3 times its width and use the short dimension as the length unit, then $\mu = 3$ and $\sigma = 8$, so that μ/σ^2 is 3/64 rather than 4/64 as it was for the square. If we use the long dimension, $\mu = 1/3$ and $\sigma = 8/3$.

This description assumes that the boundary shape and the placed shape are the same (e.g., circle). The formulation is easily adjusted for a boundary shape which differs from the placed shape, or for a launching scheme (sec. 4).