Joint Mathematics Meetings 2015 — San Antonio

An Algorithm for Creating Aesthetic Random Fractal Patterns

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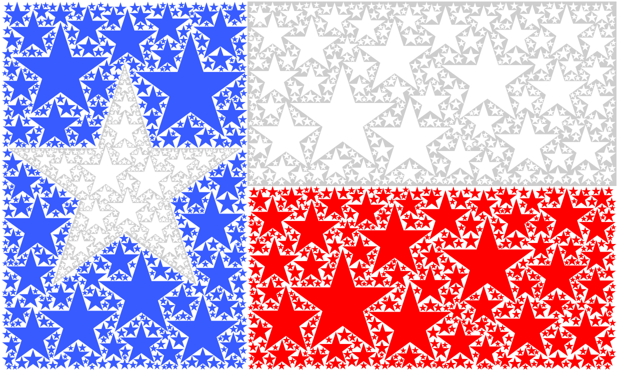


Outline

- Background and the “Area Rule”
- The algorithm
- A conjecture
- Sample patterns
- Wallpaper patterns
- Spread out patterns of circles
- A 3D pattern
- Conclusions and future work
- Contact information

Background

Our original goal was to create patterns by randomly filling a region R with successively smaller copies of a motif, creating a fractal pattern.

This goal can be achieved if the motifs follow an "area rule" which we describe in the next slide.

The resulting algorithm is quite robust in that it has been found to work for hundreds of patterns in (combinations of) the following situations:

- The region R is connected or not.
- The region R has holes — i.e. is not simply connected.
- The motif is not connected or simply connected.
- The motifs have multiple (even random) orientations.
- The pattern has multiple (even all different) motifs.
- If R is a rectangle, the pattern can be **periodic** — it can tile the plane as one of the 17 **wallpaper patterns**. Triangular regions R can generate other wallpaper patterns.

The Area Rule

If we wish to fill a region R of area A with successively smaller copies of a motif (or motifs), it has been found experimentally that this can be done for $i = 0, 1, 2, \ldots$, with the area A_i of the i-th motif obeying an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

The Algorithm

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

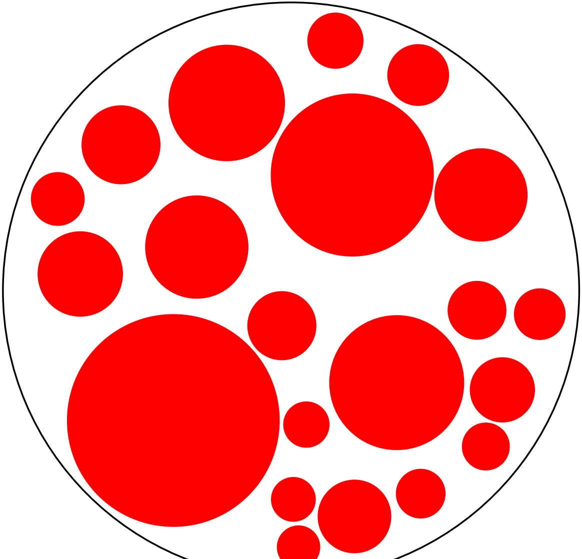
We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

In the following slides, we show a simple example of how this works for a circular region filled with circles as motifs.

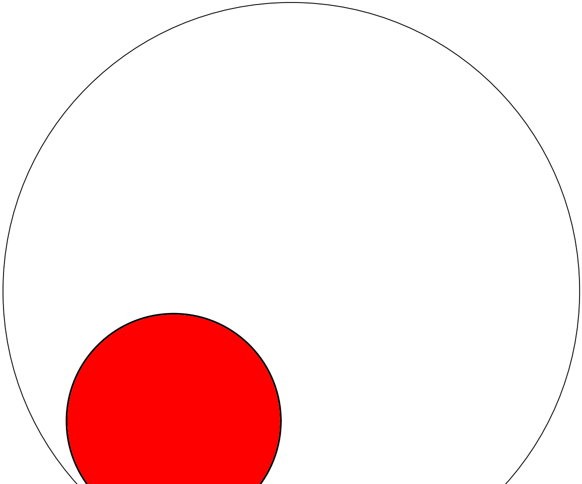
A pattern of 21 circles partly filling a circle

A circle region partly filled by 21 circle motifs

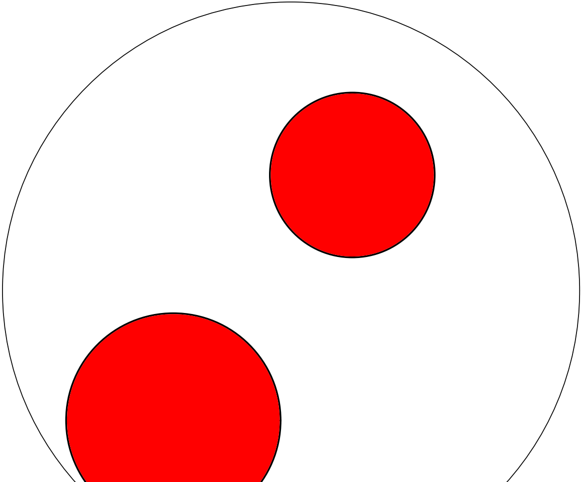
(Note: $c = 1.30$ and $N = 2$ in this example)



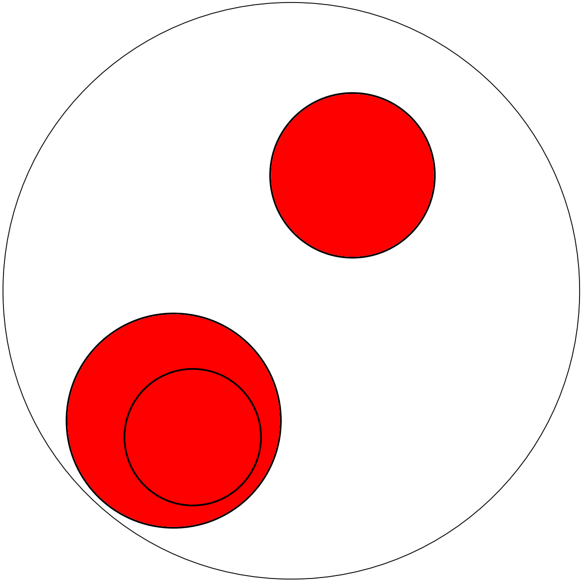
Placement of the first motif



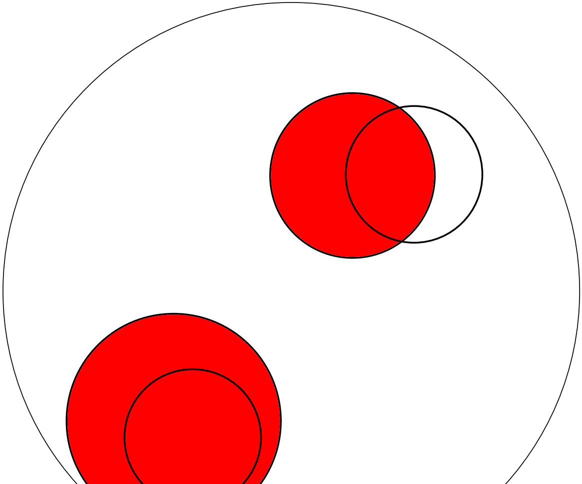
Placement of the second motif



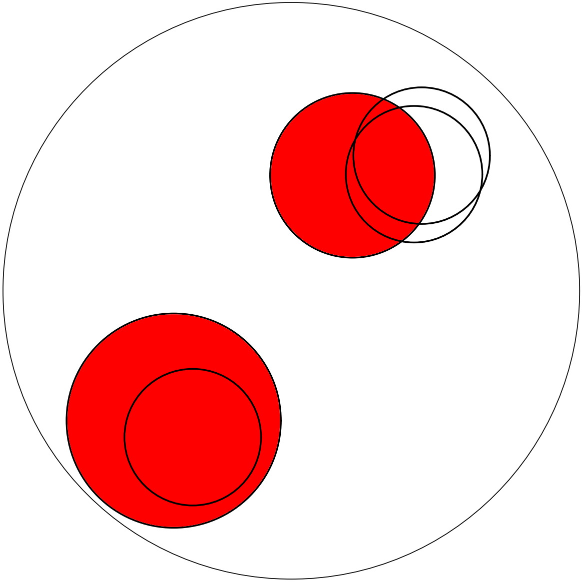
First trial for the third motif



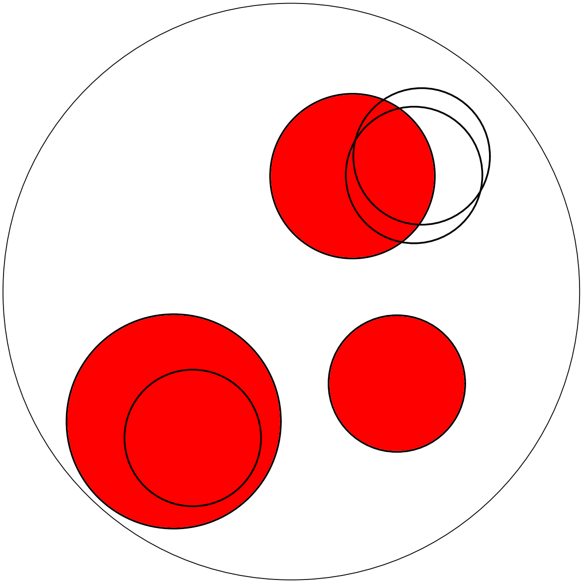
Second trial for the third motif



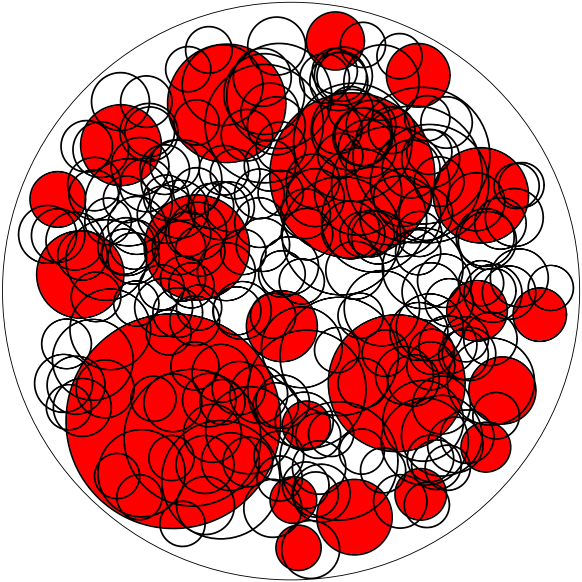
Third trial for the third motif



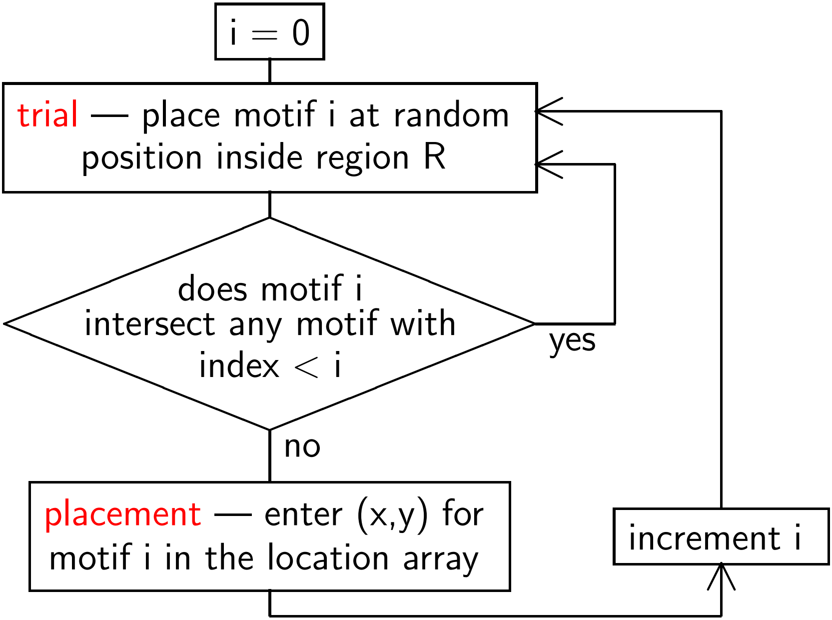
Successful placement of the third motif



All 245 trials for placement of the 21 circles



A Flowchart for the Algorithm



A Conjecture

Conjecture: For reasonably defined (i.e. not pathological) regions R and motifs, the algorithm will not halt for values of c and N within a subset of the $cN-plane$ satisfying $1 < c < c\_max$ and $N > N\_min > 0$, for appropriate values of c_max and N_min (which depend on the shapes of R and the motifs).

Typically values of c_max seem to be somewhat less than 1.5; often the values of N that were used were 2 or greater (not necessarily integer).

This algorithm has been implemented in dimensions 1, 2, 3, and 4, though we note that 1D patterns are not very interesting, and the "front" motifs in 3D and 4D obscure the motifs behind them.

In 1D, in which the motifs are line segments, it has been proved that the algorithm never halts for any c with $1 < c < 2$.

Also, the fractal dimensions of the patterns (not the unused portion of R) can be calculated to be $1/c, 2/c$, and $3/c$ in the 1D, 2D, and 3D cases respectively, which leads to another conjecture that the fractal dimension is d/c in d-dimensional space.

Dependence of patterns on **c** and **N**

By examining the formula that gives the Area Rule:

$$A_i = \frac{A}{\zeta(c,N)(N+i)^c}$$

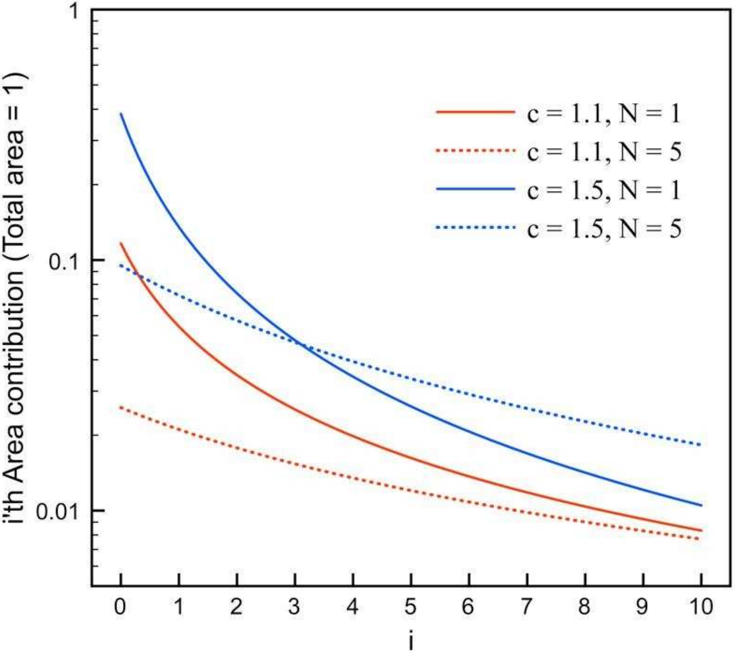
one can see that as c increases or N decreases, there is a larger difference in the sizes of the first few motifs.

Conversely, as c decreases or N increases, the first few motifs are closer in size.

The next slide shows a graph of how the sizes of the i-th motif decrease for different values of c and N.

Following that, we show how patterns depend on c and N.

Graph of areas A_i for different values of c and N

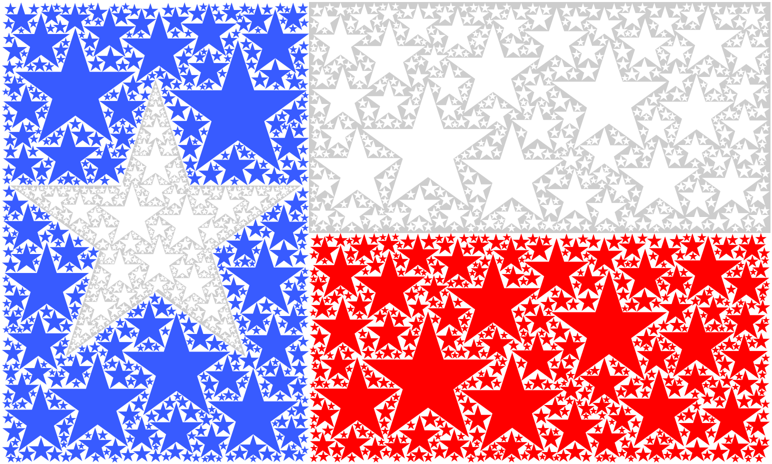


Sample Patterns

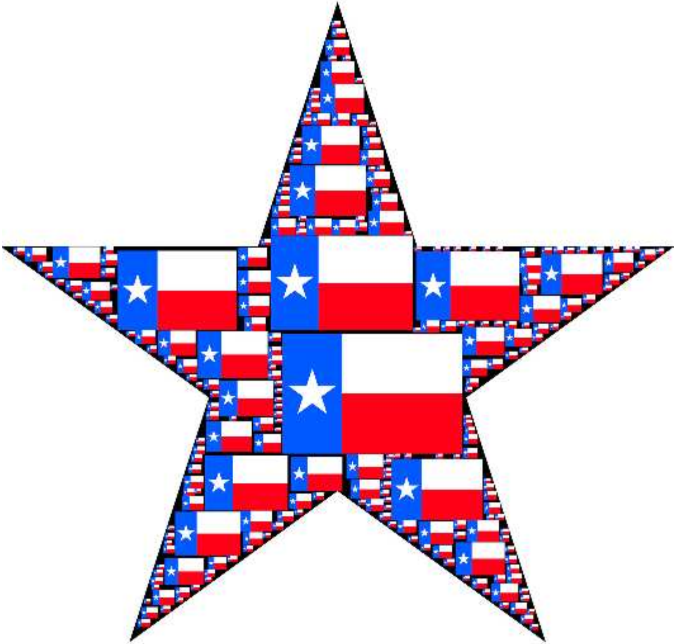
In the following slides, we exhibit the robustness of the algorithm by showing combinations of:

- Connectivity of the bounding region R.
- Non simply connected regions R.
- Non connected or non simply connected motifs.
- Motifs with multiple or even random orientations.
- Multiple, even all different, motifs.
- Periodicity for rectangular regions R.

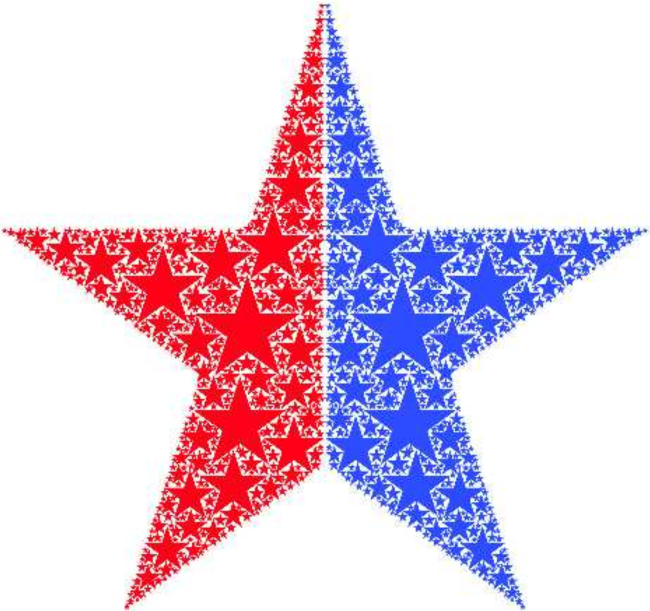
The Texas flag made up of non-lone stars



A star made up of Texas flags

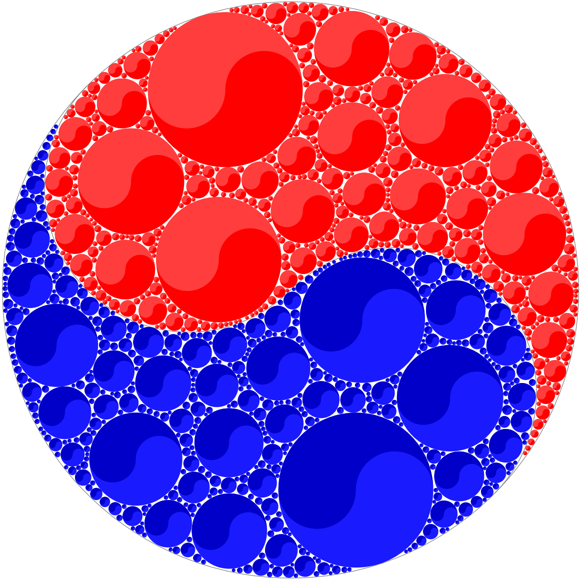


A star made up of stars with 2-color symmetry



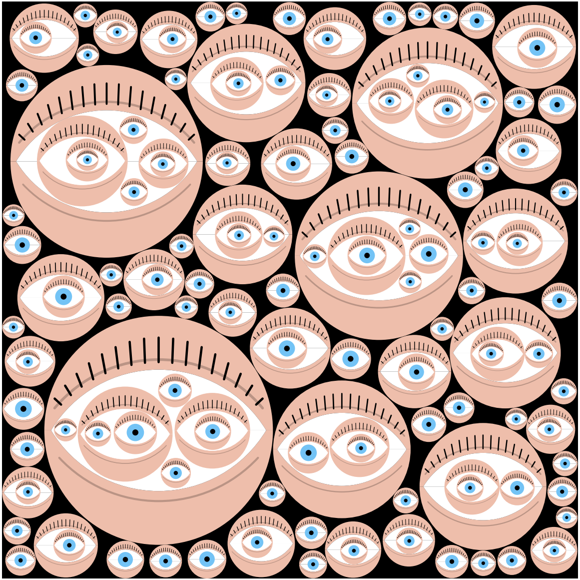
Two regions forming a yin and yang

In this pattern, $c = 1.47$ and $N = 3$, with 92% fill;
it has 180° rotational color symmetry.



A pattern of non-simply connected eye motifs

In this pattern, $c = 1.20$ and $N = 3$, with 56% fill; only eyes with no contained eyes have pupils.



A periodic (p1) pattern of randomly oriented peppers

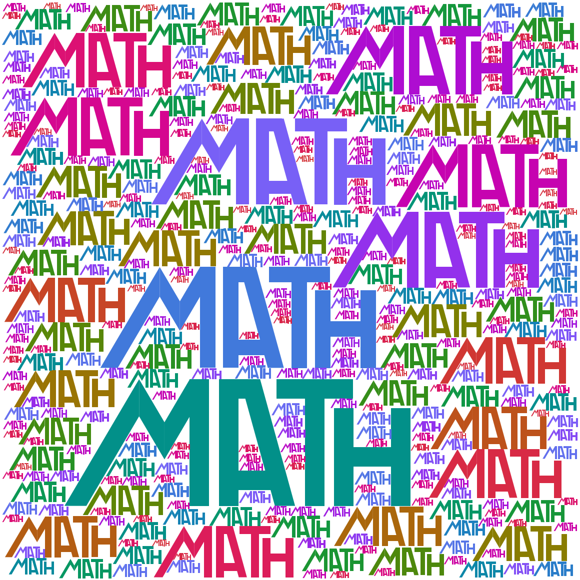
In this pattern, $c = 1.26$ and $N = 3$ with 80% fill.

A pattern with the word ART as a motif

In this pattern, $c = 1.15$ and $N = 3$ with 53% fill.

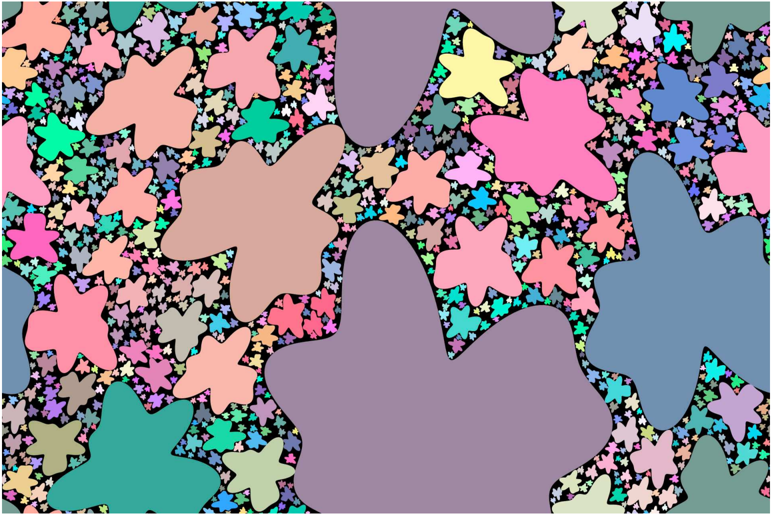
A pattern with the word MATH as a motif

In this pattern, $c = 1.26$ and $N = 2$ with 50% fill.



A periodic (p1) pattern of different random blobs

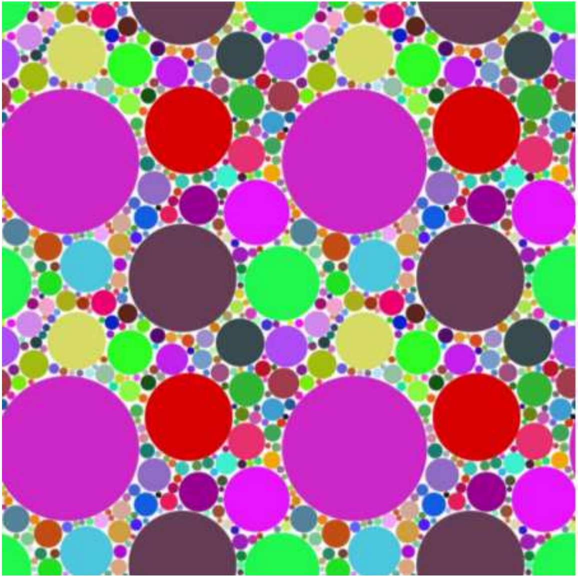
In this pattern, $c = 1.23$ and $N = 1$ with 82% fill.



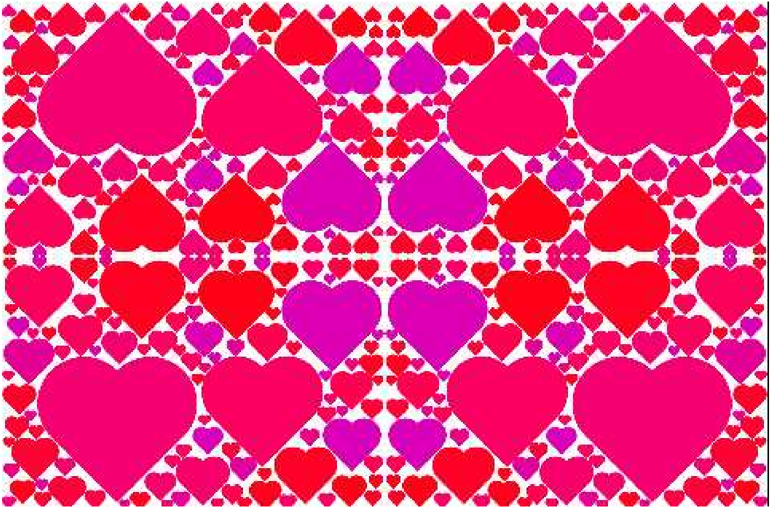
Patterns with Wallpaper Symmetry

- The previous pattern and the randomly oriented pepper pattern can tile the plane. In crystallographic terms, they have $p1$ symmetry (or **o** symmetry in orbifold notation), the simplest kind of "wallpaper" symmetry. The next pattern is another example.
- We can also create wallpaper patterns based on the other 16 possibilities. By filling the fundamental region of a wallpaper pattern with randomly placed motifs, one obtains an aesthetic combination of small-scale randomness with large-scale regularity.
- We have concentrated on wallpaper patterns that have mirrors as the boundaries of their fundamental regions, including $p2mm, p4mm$, and $p6mm$, which are respectively also denoted by their short forms, $pmm, p4m, p6m$, or by their orbifold notations $*2222, *442, *632$. For these patterns, we can either avoid putting motifs on the mirrors or center motifs with bilateral symmetry on the mirror boundaries.

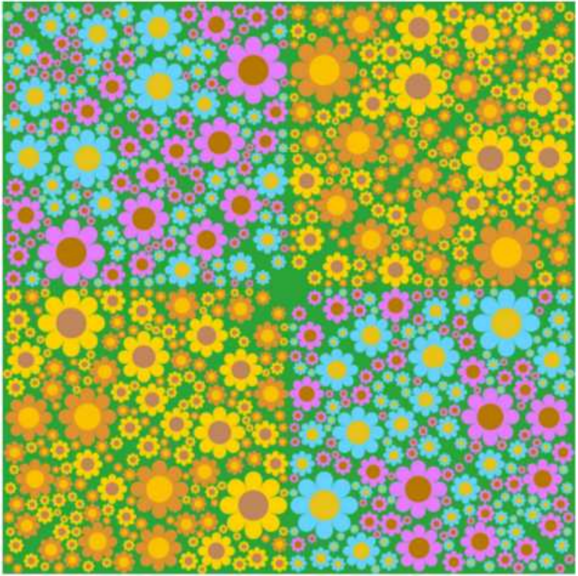
A pattern of circles with p1 symmetry



A pattern of hearts with p2mm (*2222) symmetry

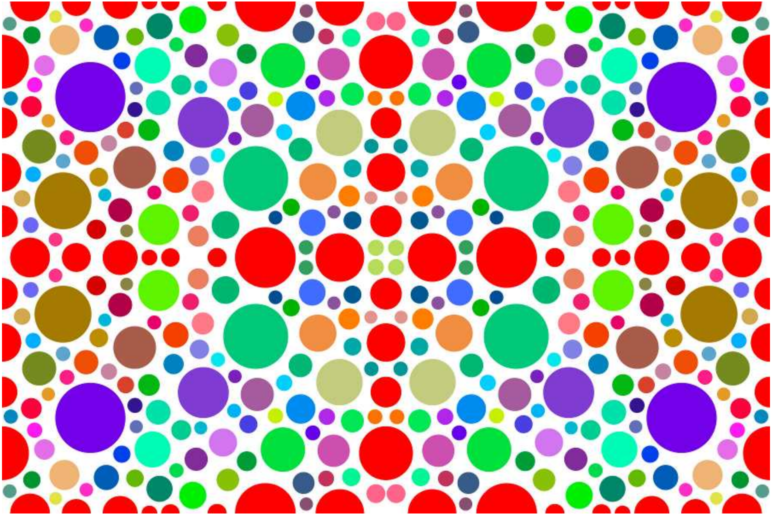


A pattern of flowers with p4mm (*442) 2-color symmetry

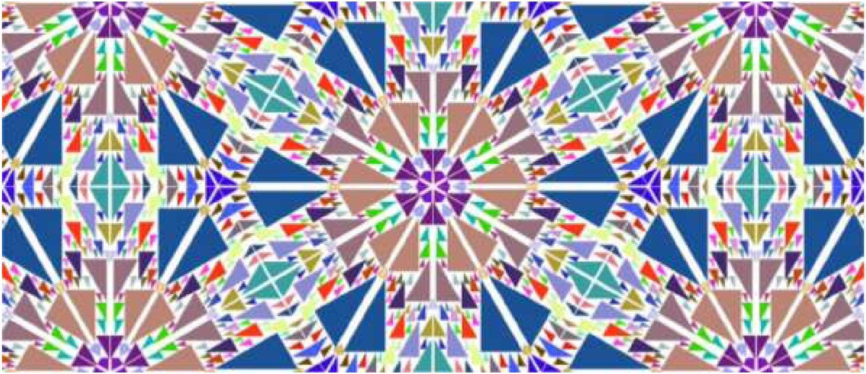


A Rorschach pattern of upright white squares and tilted black squares with p4mm (*442) 2-color symmetry

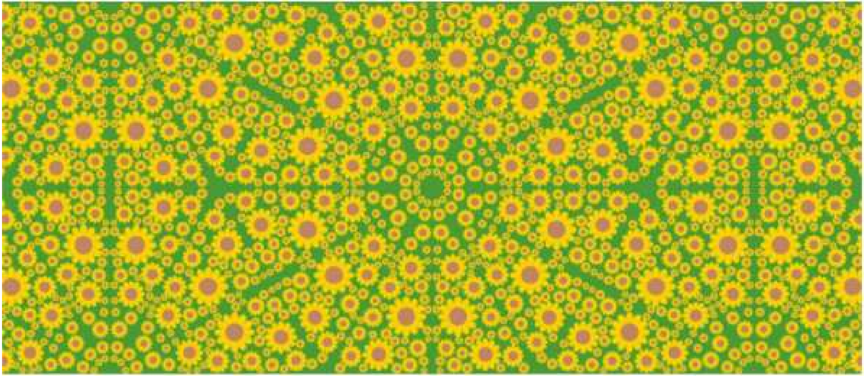
A circle pattern with p4mm (*442) symmetry with red circles on the mirrors



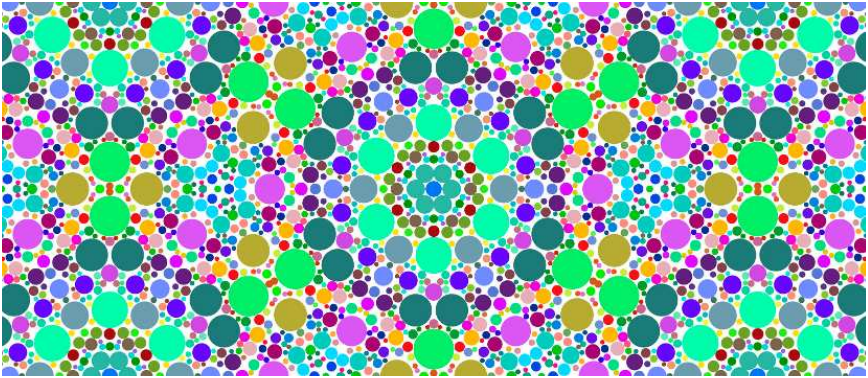
A pattern triangles with p6mm (*632) symmetry



Another flower pattern, but with p6mm (*632) symmetry



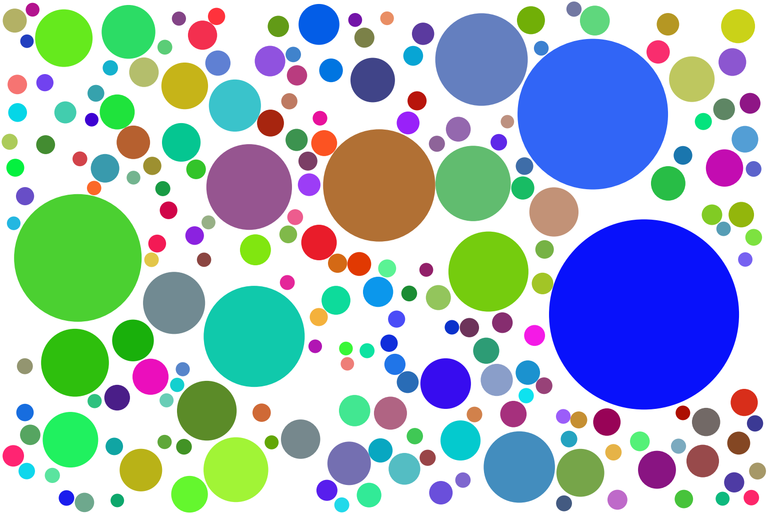
A circle pattern with p6mm (*632) symmetry and circles on the mirrors



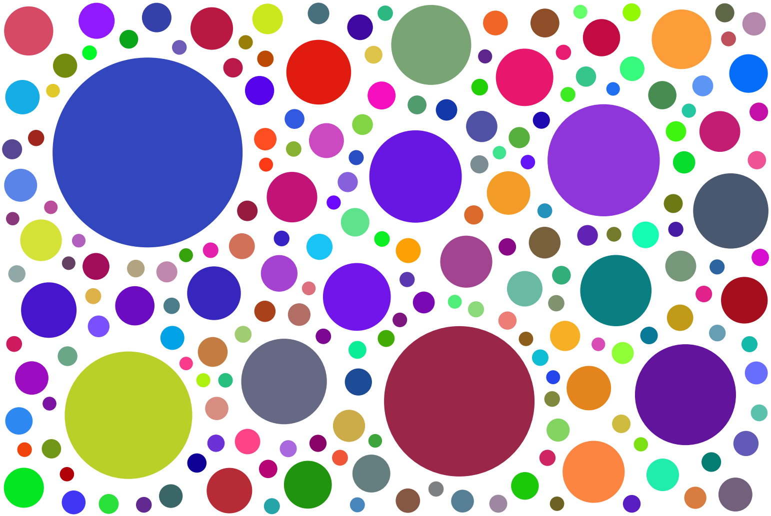
Spread out Circle Patterns

- In what we have considered so far, we only required that a new motif did not overlap any previously placed motifs. With circle motifs, which means that the center of the new circle is at least its radius from any previous circle. The algorithm seems to work even if we require that the new circle be farther away, even twice its radius from any previous circle.
- We use β to denote the ratio of the distance from the center of the new circle to any previous circle divided by the radius of the new circle.
- The next slide shows an ordinary placement of circles with $\beta = 1$.
- The following slide shows a placement of circles of the same radii, but with $\beta = 2$.
- As might be expected, the number of trials needed to obtain a successful placement increases considerably as β increases from 1.0.

An ordinary circle pattern with $\beta = 1$

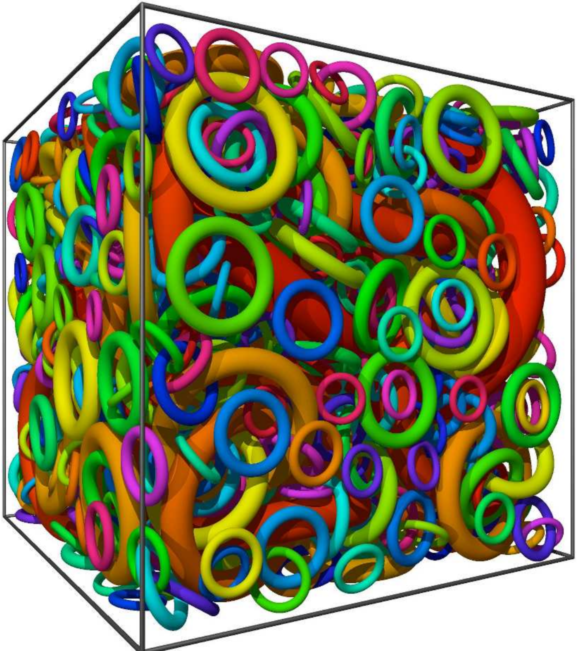


A pattern of the same circles, but with $\beta = 2$



A 3D pattern of tori by Paul Bourke

Note that some tori are linked.



Future Work

- Since the algorithm seems to be so robust, it would be reasonable to test it with new combinations of 2D regions and motifs.
- Though 1D patterns are not very interesting, and the motifs of 3D patterns block views of the interior, still there may be interesting 3D patterns to be discovered.
- We have displayed two patterns that are periodic, and thus tile the plane. Such a tiling would have the simplest plane symmetry group $p1$ (or **o** in Conway notation). We have made some progress in creating locally fractal patterns having global symmetries of the other 16 plane symmetry groups using our techniques, but there is still more work to be done.
- There are a few things that can be proved mathematically about these patterns, but there are a number of conjectures that have yet to be proved — such as the non-halting of the 2D algorithm for reasonable values of c and N.

Acknowledgements and Contact

We would like to thank Paul Bourke for his contributions to fractal geometry and for his 3D work in particular. His web page is
http://paulbourke.net/

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