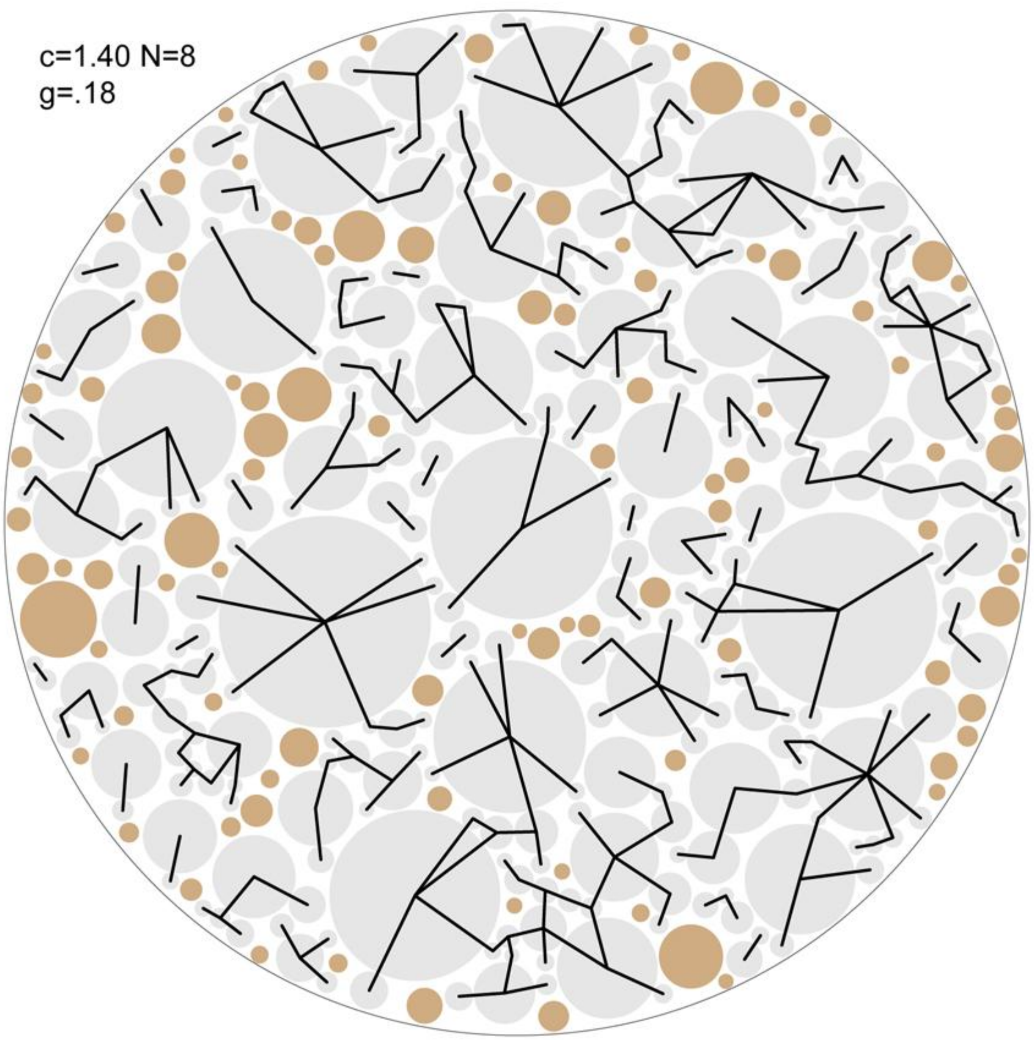
Proximity Tree Statistics in 2D Statistical Geometry Fractals.

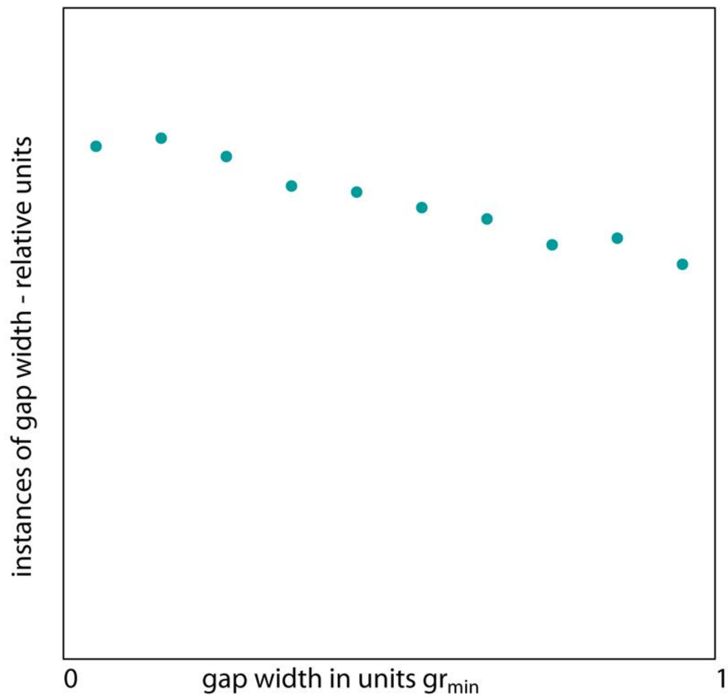
1. Example.

Figure 1 shows an example of a statistical geometry fractal, in which circles are fractalized within a circular boundary. It is natural to inquire about the near neighbors in such a fractal. The *proximity tree* is one way of studying this question.

In constructing the proximity tree the perimeter-to-perimeter gap is computed for each circle pair, and a line drawn if it is below a value gr_{min} where g is a parameter and r_{min} is the smallest circle radius. It can be seen that this creates a number "nets" which are mutually linked but not linked to the rest of the tree. (This is to be viewed as the definition of "net"). The darker tan circles are unused -- they have no gaps short enough.

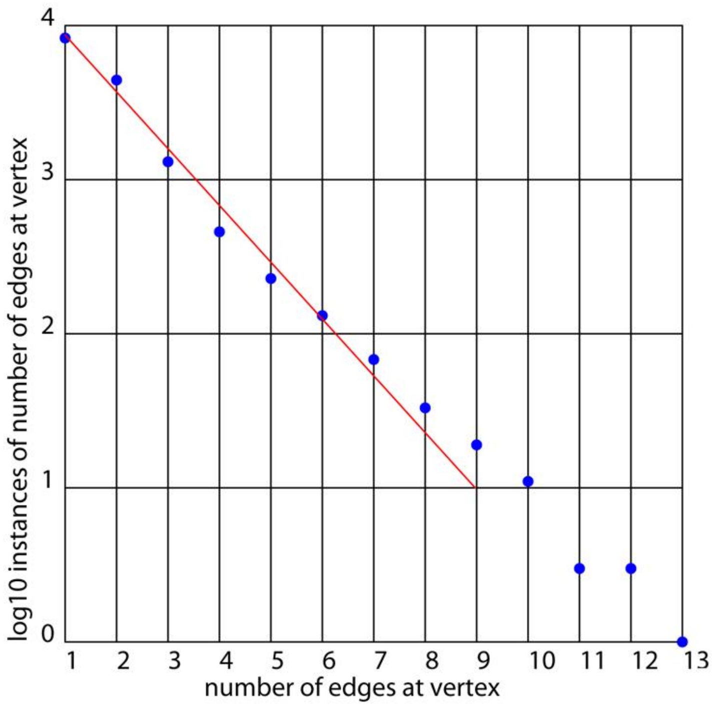


2. Statistics 1 -- The Gap Distribution



This is a histogram of all the gap widths (perimeter-to-perimeter distances) below gr_{min}. It can be seen in this plot using linear coordinates that the probability of a given gap width falls by a moderate amount as the gap width increases. This is typical, and varies only slightly with the parameter set c, N, g.

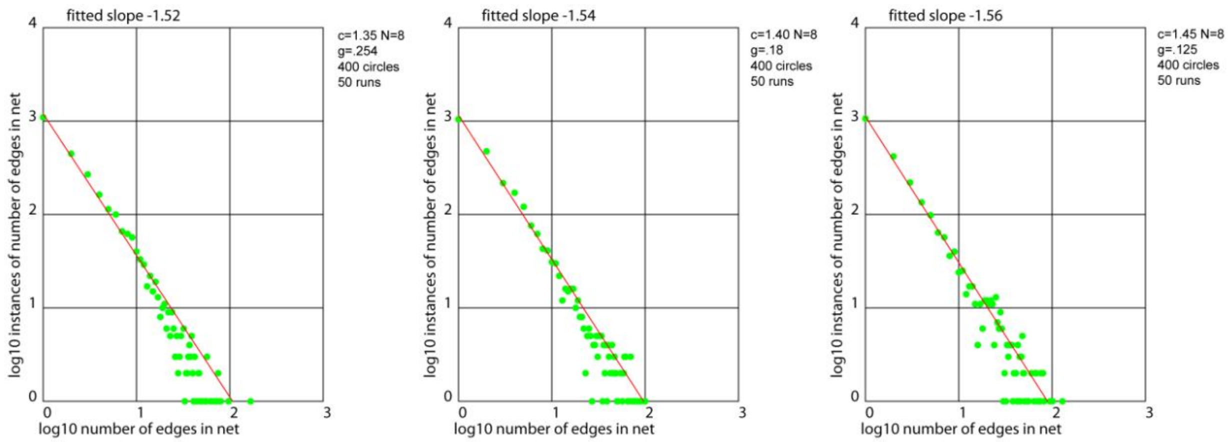
3. Statistics 2 -- The Edges per Vertex Distribution.



A number e_v of edges meet at each vertex in the graph. This number varies from 1 to some much larger value. Fig. 3 shows the observed distribution in semilog coordinates. The red line is a weighted least squares regression, and is a good fit. This indicates that the number of edges per vertex drops off with e_v as a *decaying exponential function*. This decaying exponential was seen for N larger than 5-6-7 (where the first few circles placed do not have large size changes) but a more complicated behavior holds for lower N. In the instance shown, e_v drops by a factor e as the number of edges increases by about l.16 according to the weighted least-squares regression line.

Intuition might suggest that e_v should go on average as the perimeter of the circle, which would result in a power law with exponent c/2 rather than what is seen here.

4. Statistics 3 -- The Edges per Net Distribution.



Histograms are shown for c values 1.35, 1.40, and 1.45. In each case the parameter g has been adjusted *so that the average number of edges in the proximity diagram is about 300* (for 400 circles). The substantial scatter in the data near the bottom of the diagrams arises from sampling error. Data has been accumulated from 50 runs in order to improve accuracy. The *average edge number* is the total number of edges created in all 50 runs, divided by 50.

The net sizes (the number of connecting edges e_n) have been extracted, and their distribution is shown in the log-log plot of Fig. 4. It can be seen that this distribution is well approximated by a *negative power law*, i.e., the distribution of net sizes is "scale free" and in a sense "fractal". The regression line (red) was found by least squares, weighting each point by the number of instances. Power law exponents k in the area -1.5 are seen here, depending somewhat on parameters.

It is observed that the majority of the data points fall below the red line for large nets, indicating that the probability of creating very large nets falls somewhat below the power law. This behavior becomes more pronounced for lower g values (lower total edges/circles placed).

With c=1.35 runs were made for g values 0.12 (130 average edge number), 0.24 (250 average edge number), and 0.36 (350 average edge number). The corresponding exponents k were -1.84, -1.55, and -1.39.

5. Discussion.

Circles are studied here as they are mathematically the most tractable form of statistical geometry fractal. In principle one could construct proximity trees for a wide variety of shapes. The trees have a large number of parameters -- c, N, g, and number of circles placed. There may be other interesting cases and limits beside the ones treated here. The number of situations that could be studied by a diligent investigator is very large.

These trees are random, but it should be kept in mind that it is a highly constrained randomness.

The form of the proximity tree varies strongly with g (one draws an edge if the gap is less than gr_{min}). For small g values there will be very few edges and most of the circles will be unlinked. For large g values the entire fractal will be mutually linked. The view taken here is that the intermediate case where only a fraction of the circles are linked is the most interesting one. The forms of the histograms (decaying exponential for edges per vertex and power-law for edges per net) persist over a substantial range of g; the fitted slopes vary with g.

The observation that the edges per net distribution is a negative power law is perhaps the most interesting finding.

What sorts of situations might be modeled by this construction? The gaps might be thought of as barriers for the passage of some property from one circle to another, and the nets as regions possessing this property. If the gap is small enough, the property passes, otherwise not. The passage of species between a set of distinct regions could be an example.

6. References.

[1] "Fractals: Form, Chance and Dimension", Benoit Mandelbrot (1977). The *opus magnus* of fractals.