The Sevenhole -- An Example of a Very Sparse Shape

1. Introduction

In statistical geometry[1] it is found that the main property that depends on the details of the shape being fractalized is the maximum c value. Compact shapes such as squares and circles have high c values (*circa* 1.5). For studies of sparse shapes with a low maximum c it is desirable to have a shape which has a high perimeter-to-area ratio and an overlap test which is "exact" in the sense that its accuracy is only limited by the precision of the floating-point numbers used in the computations.

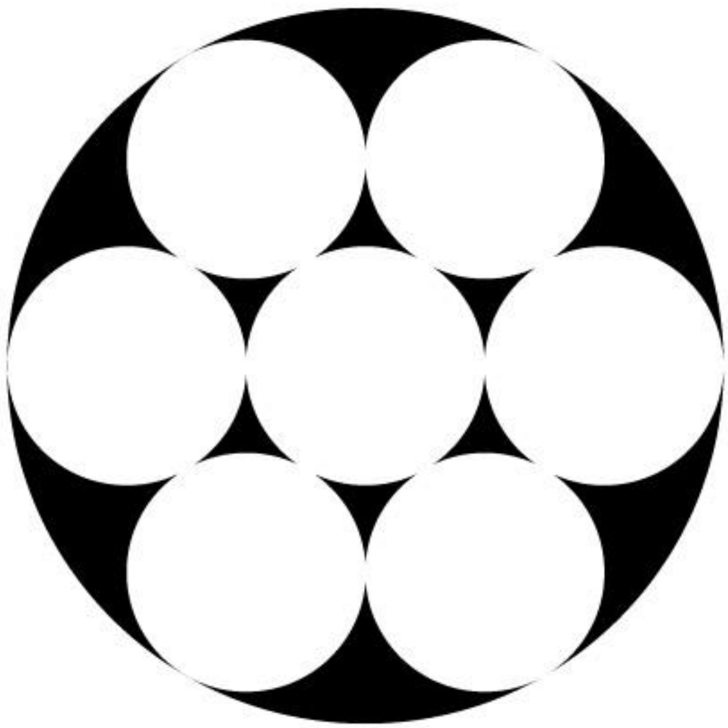


Fig. 1. The sevenhole shape (black).

This is a sparse shape well suited to computational studies. While it is readily defined in terms of circles, it is not a classic geometric shape with an established name, and it is here simply labeled *sevenhole*. It consists of a disc of radius R, with 7 mutually tangent round holes in it, each of radius $R/3$. The shape is highly symmetric.

The area is $A = \frac{2}{9}\pi R^2$ and the perimeter is $p = \frac{10}{3} 2\pi R$

Early studies show that the maximum c value is around 1.07. With such a low c value it is quite difficult to get a high percentage fill.

It can be seen that it is a member of a "family" of discs with 2, 3, 4, 5, or 7 symmetrically-arranged holes in them.

[1] Those not familiar with the statistical geometry algorithm should read the Shier-Bourke paper cited below under "References".

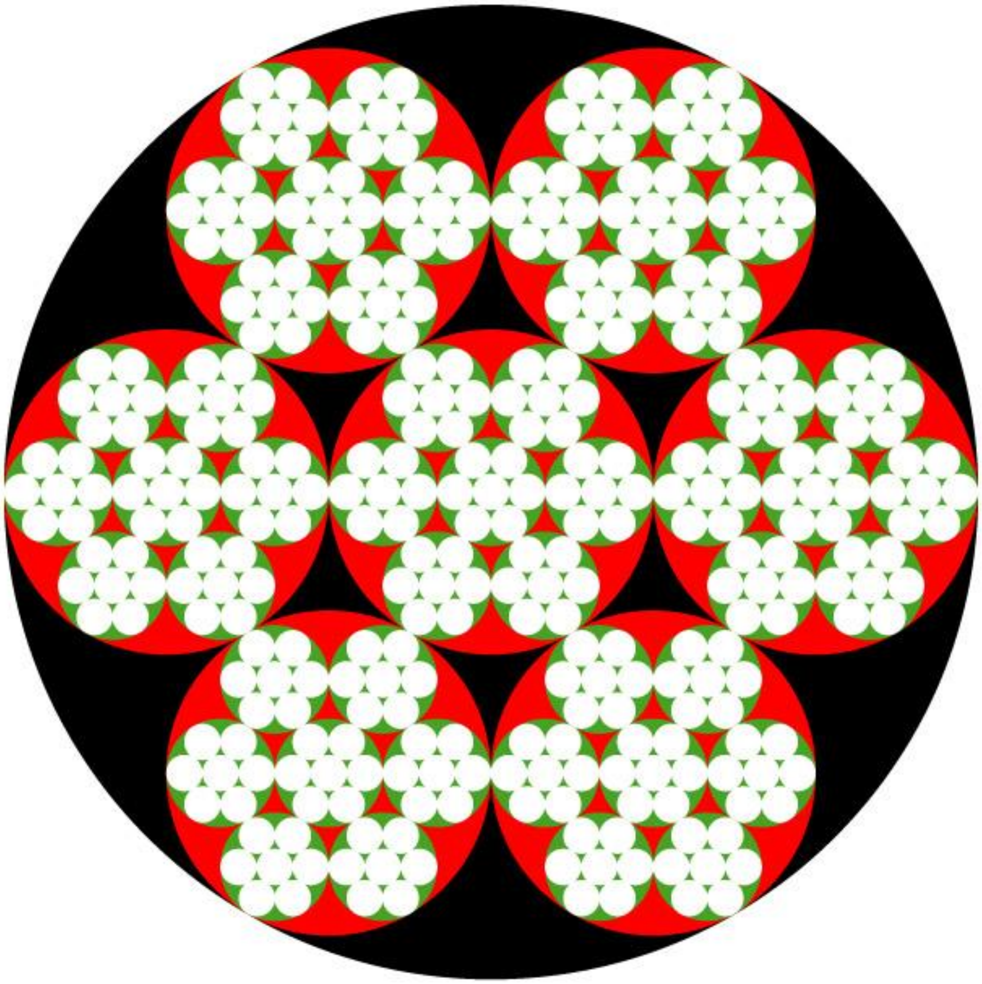


Fig. 2. An additive recursive fractal based on the sevenhole. The first level of recursion is in red, and the second in green.

The sevenhole is the basis for the recursive fractal[2] shown in Fig. 2. A smaller shape with radius 1/3 of that for the previous generation is placed in each hole. At each recursion 7 new shapes are added, each with linear dimensions 1/3 of the previous recursion. By the usual rules for recursive fractals the fractal dimension D is given by $D = \log(7)/\log(3) = 1.771$. The result of such a process "in the limit" is a completely filled circle.

The sevenhole has very sharp cusps where the circles touch, and regions of vanishing width. The recursive form is illustrated in more detail in Appendix A.

2. Computation

Numerical computation is simpler for this shape than for most sparse shapes, and is fast. The c value used, 1.071, is such that perhaps half of the runs have early failure, i.e., this value is above c_1 as defined in the Shier-Bourke paper (see references below).

[2] We assume an additive process here. Most mathematics texts on fractals would consider a full circle with the shapes of Fig. 2 *et seq*. successively deleted from it. Physics papers use an additive procedure.

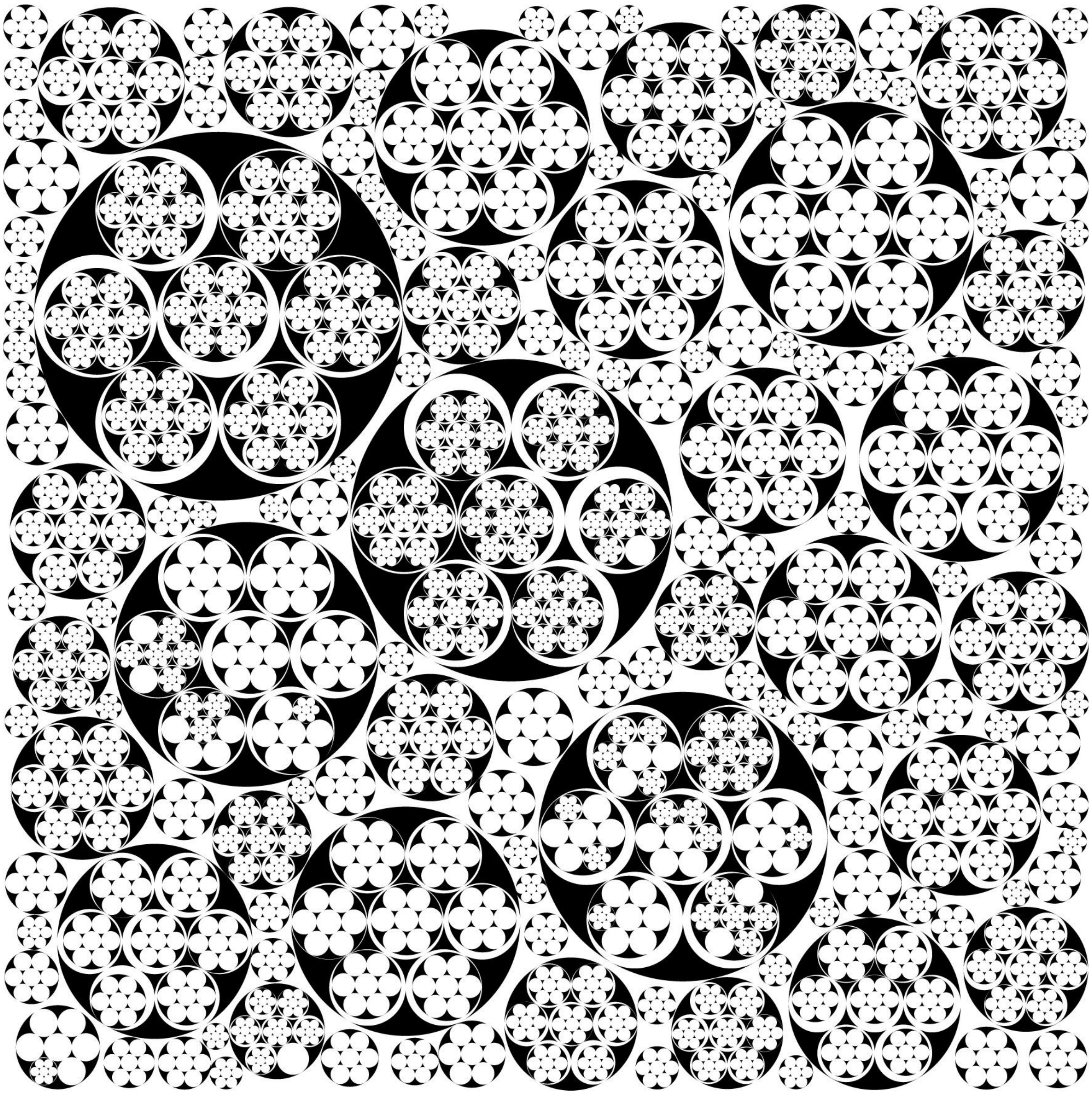


Fig. 3. The sevenhole fractalized with c=1.071, N=3.5, 500 shapes, 30% fill.

A random fractalization of the sevenhole is shown in Fig. 3. If you look at the largest first-placed shape you see a strong resemblance to Fig. 2, except that the smaller shapes don't completely fill the holes because of the randomness of the algorithm. We can speak of *pseudo-generations*. The second-largest shape is filled to 2 pseudo-generations with only one empty hole, while the third-largest is about half-filled in the 2nd pseudo-generation. The fourth-largest shape has only a few shapes of the 2nd generation. The next-smaller shapes have only the 1st generation. In general most of the shapes are close to a complete generation of filling. The fractal dimension D is $D = 2/c = 2/1.071 = 1.87$.

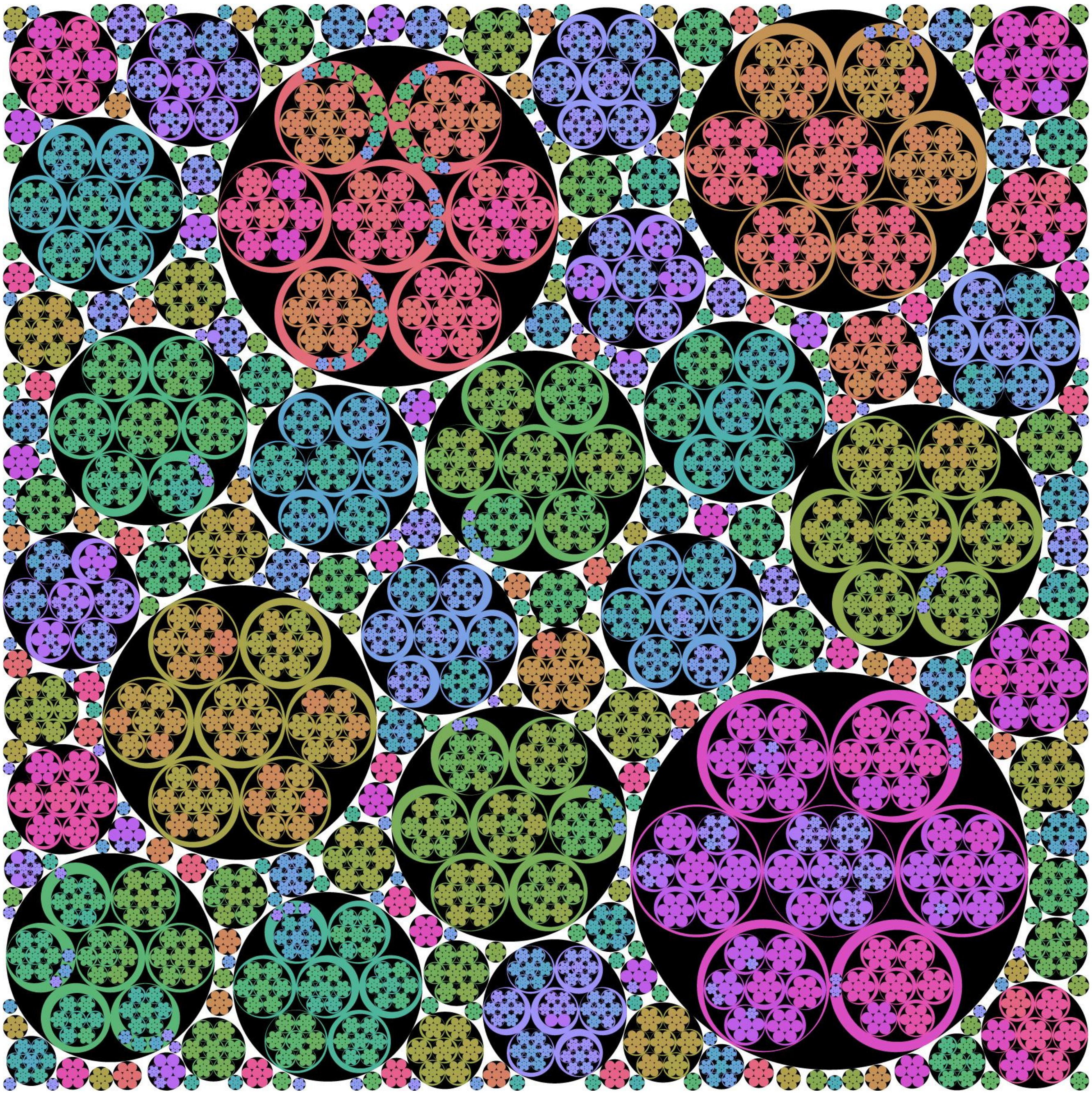


Fig. 4. The sevenhole fractalized with c=1.071, N=3.0, 2100 shapes, 38% fill. The color scheme is log-periodic.

An example with log-periodic color is shown in Fig. 4. There are two ways to draw the shape. It can be drawn with circular arcs which form a single contour with the shape of the edge of the black region of Fig. 1, or one can draw a black circle with seven circles within it. The latter choice has been made in Fig. 4. If we define a quantity $u = R_j / R_0$ (R_j is the radius of the j-th shape) then the hole color is periodic in $log(u)$ and repeats when u decreases by about a factor of 2.8.

The color begins with magenta (largest shape) and repeats with nearly the same color in the first and second generations. A periodic sequence of magenta → orange → brown etc. is followed. By the third generation the color within largest shape has moved off toward blue. It can be seen that the color is approximately the same within every shape, with small variations due to the randomness of the process. This color scheme brings out the high degree of order in the pattern.

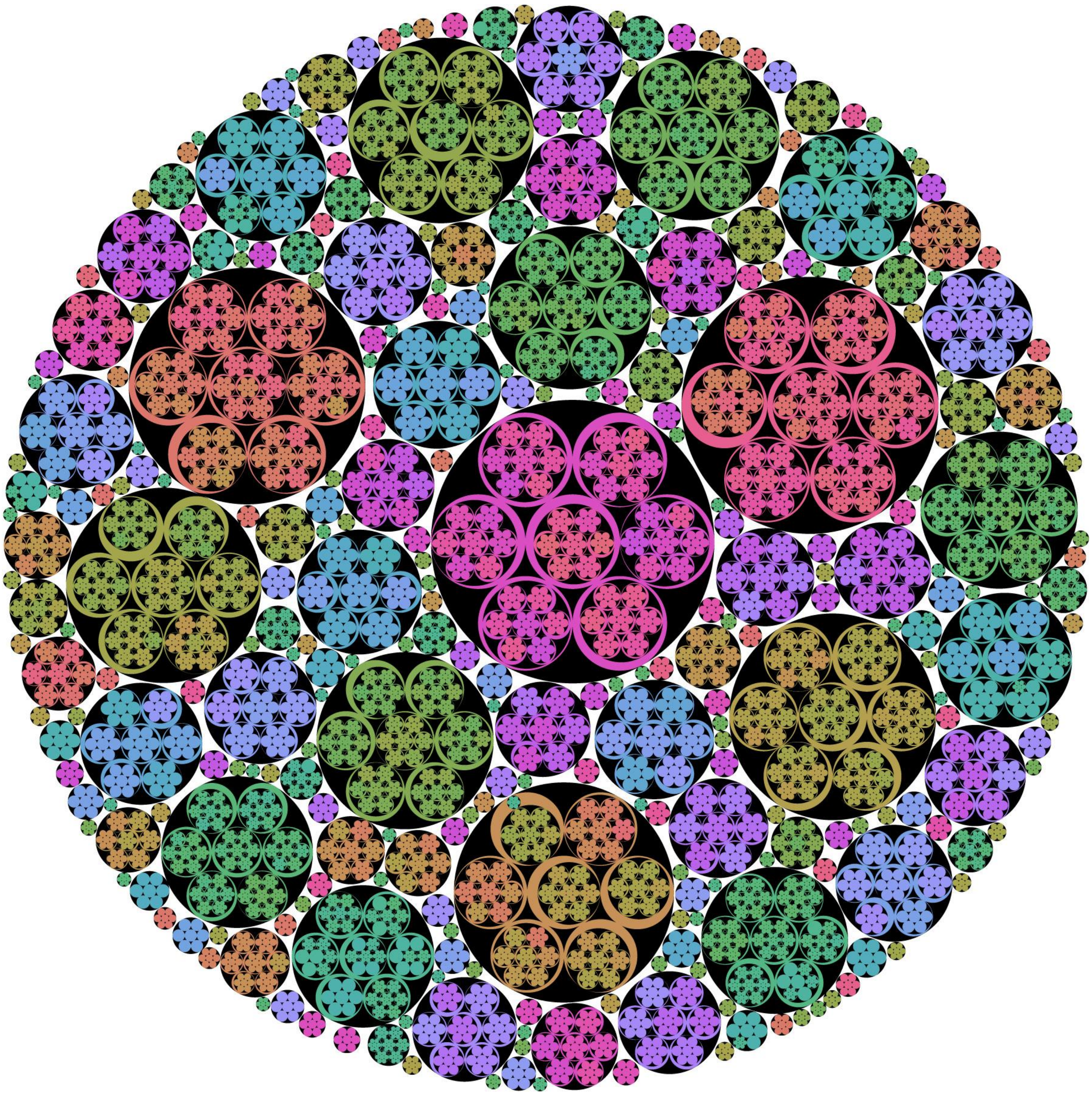


Fig. 5. The sevenhole fractalized within a circular boundary with c=1.076, N=5.0, 1500 shapes, 36% fill. The color scheme is log-periodic.

Because of the tight grouping of the successive sizes within a given shape, we regard the shapes of nearly the same size as pseudo-generations because of their resemblance to Fig. 2.

Using a lower N and a higher c with a circular boundary we get the pattern of Fig. 5. Color matching within any shape is somewhat closer than in Fig. 4, and the average "degree of fit" is somewhat better. The difference from the recursive fractal of Fig. 2 is that while a similar pattern is seen within the large shapes, at the same time there is a steady increase of new placements within the white area[3], each of which will in turn generate an ever-smaller pseudo-recursive pattern within it as the algorithm moves forward. Here $D = 2/c = 2/1.076 = 1.86$.

The close matching of the color between pseudo-generations within a given shape shows that the effect of order (due to the tight constraint of non-overlap) prevails over the randomness of the trials.

3. Comments.

The sevenhole and other members of its family with 2, 3, 4, and 5 holes are good examples for the study of statistical geometry fractals with sparse shapes. It nicely exemplifies the blending of order and randomness produced by the statistical geometry algorithm when c approaches its upper limit. The advantages for practical computation include simple formulas for area and perimeter, and an overlap test which is as precise as the resolution of the floating-point numbers used.

4. References.

The best source for a mathematical description of the algorithm is:

"An Algorithm for Random Fractal Filling of Space", John Shier and Paul Bourke
Computer Graphics Forum, Vol. 32, Issue 8, pp. 89-97, December 2013.

Copies of the last version of the paper to go to the editor can be downloaded from the author's web site (or that of Paul Bourke). The most recent publication on statistical geometry fractals is

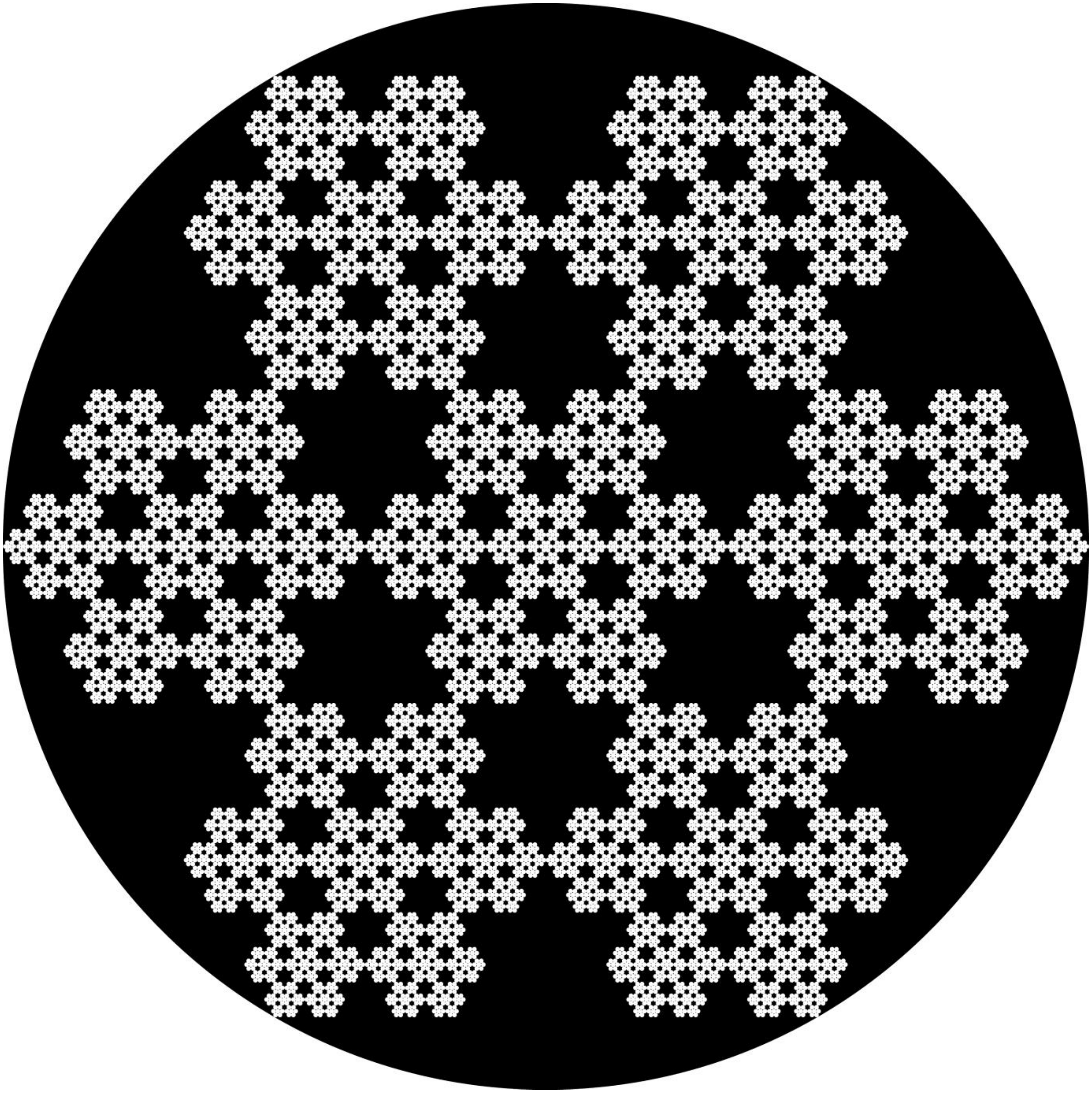
Dunham and Shier, "The Art of Random Fractals" in Proceedings of the 2014 Bridges conference, Seoul, Korea (August 2014).

It can be viewed at the Bridges web site.

[3] Not only do new shapes which are not part of an existing pseudo-recursive pattern appear in the white area, but eventually they will appear within the colored crescents created by the failure of new placements to completely fill the holes. Such shapes are visible many places in Fig. 4.

Appendix A -- The Recursive Form.

The recursive form is shown below down through the 3rd recursion.



The black holes in the image look somewhat like the triadic Koch island[4], although they differ in having boundaries made up entirely of circular arcs. The image below shows detail of the structure, from which it is seen that the black region also includes small black "islands" of a size which diminishes 3x at each recursion. As noted above the fractal dimension D is $log(7)/log(3)$. The black-white boundary will have a different fractal D. The white region(s) can be thought of as a sort of self-similar snowflake, or perhaps a snowflake composed of ever-smaller snowflakes.

[4] The Koch island can be seen in plate 39 of Mandelbrot's book "Fractals Form, Chance and Dimension" (1977) on p. 39.