Wallpaper Groups and Statistical Geometry

John Shier

Abstract. The arrangement of regions filled by statistical geometry into arrays with specific symmetry properties is studied for the space groups p2mm, p4mm, and p6mm. When the randomness inherent in the algorithm is combined with the rigid order of space-group symmetries the results can be visually appealing.

1. Introduction

Basic statistical geometry fractals [1] have no symmetry elements of the usual kind because of the randomness in the placement of the shapes. This prompts the question "Can one *impose* symmetry on these structures?" It was realized quite early that one can produce periodic rectangular tiles by placing "partner" shapes at locations $x+X_0$ and/or $y+Y_0$ (where x and y are trial positions and X_0 and Y_0 are the repeat periods in x and y) whenever a shape intersects the bottom or left boundary of the region to be filled. Figure 1 illustrates this with 4 periodic tiles abutted. Such an arrangement has translational symmetry, but no other symmetry, and belongs to a quite simple wallpaper group [2]. Such tiled arrangements are required for many decorative purposes, such as fabric or wrapping paper.

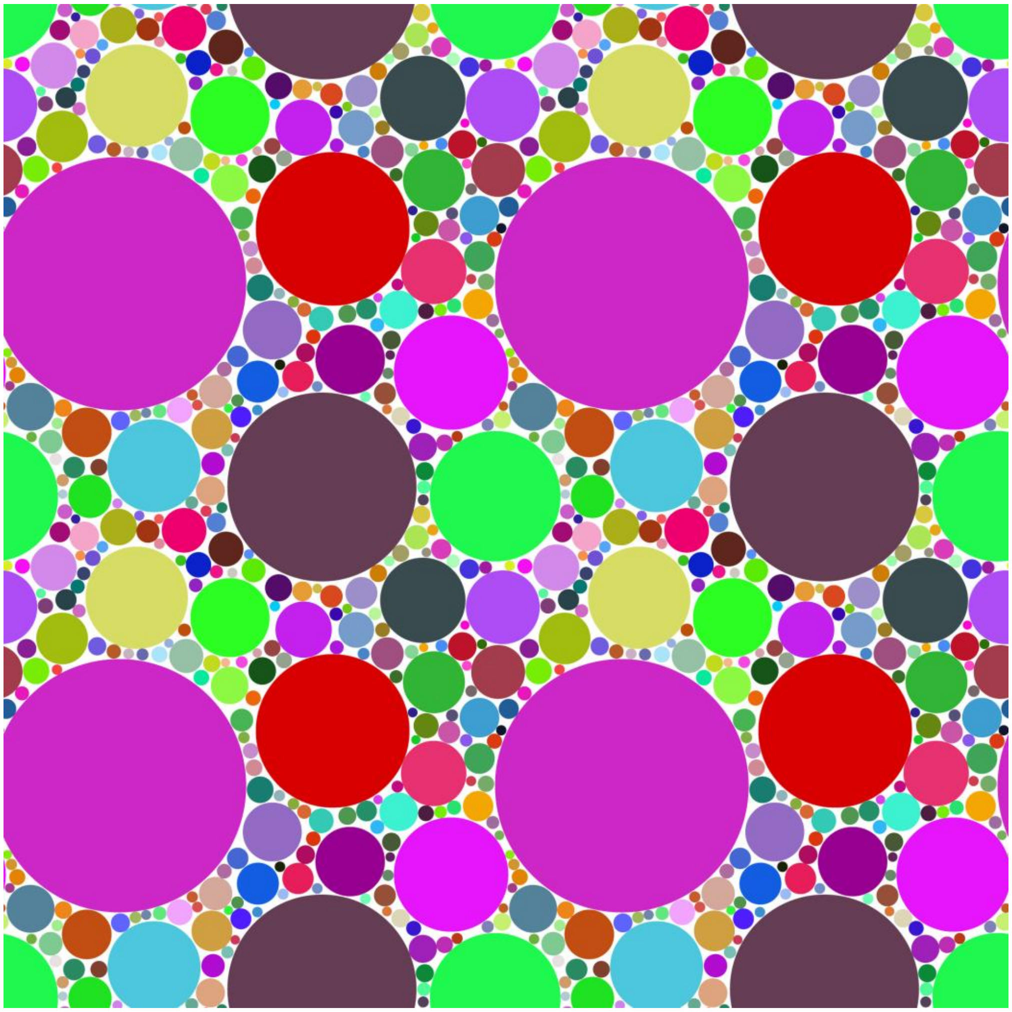


Fig. 1. An illustration of the use of periodic boundaries to produce a tiled pattern.

How could one extend this to higher-order symmetries? What about p4mm for example, with its many mirror lines? If we think of circles, they must either be nonintersecting with the mirror lines, or exactly centered on the mirror lines. The case of nonintersection with the mirror lines is the simplest and will be further explored in what follows. The combination of random placement and exact positioning on the cell boundary requires an extension beyond previous work.

Statistical geometry fractals are quite random (with the degree of randomness controllable to some extent by choice of the parameters *c* and *N*). The imposition of the symmetry requirements of p4mm or p6mm creates a highly ordered pattern with a random primitive unit cell. It thus represents a balance or competition between order and disorder, which in the author's opinion is a desirable thing in art.

2. p2mm Symmetry.

Patterns having this symmetry are readily achieved with any rectangular image with inclusive boundaries. Figure 2a illustrates the concept with a heart pattern. It is only necessary to mirror-and-repeat as shown; this can be done either by computation or with suitable photograph processing software.

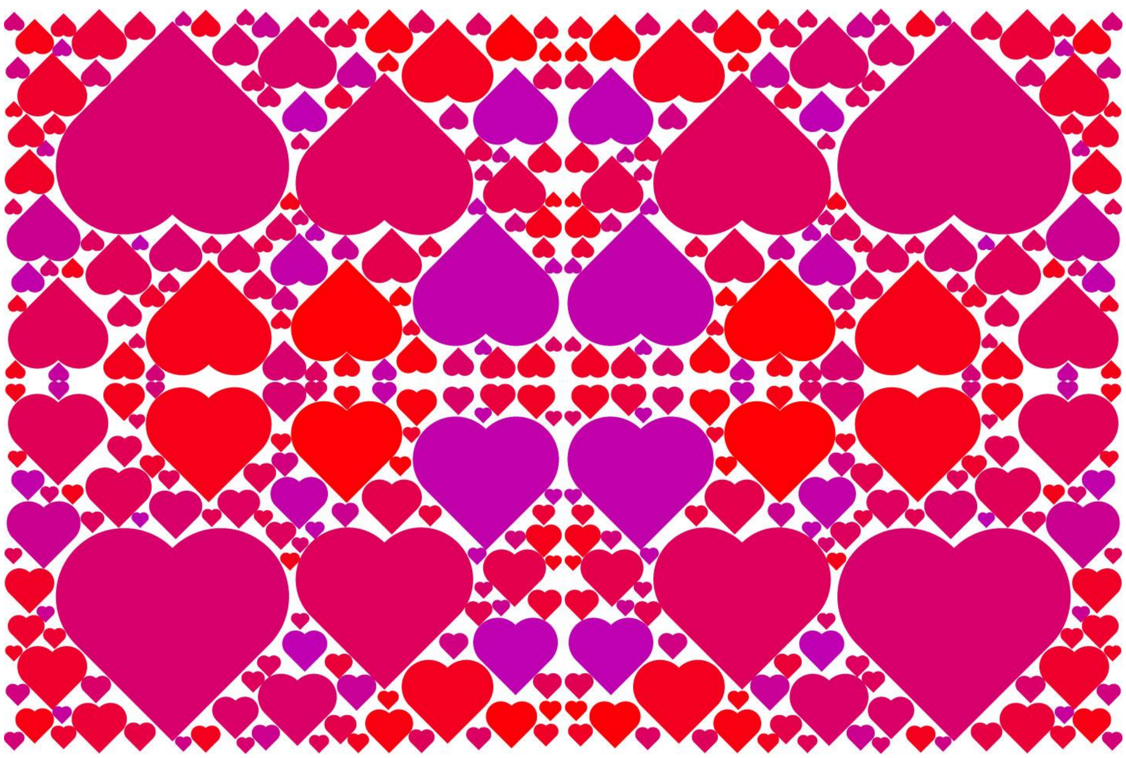


Fig. 2a. A p2mm symmetric array of hearts created by mirror-and repeat of an image which has inclusive boundaries.

The requirement that shapes cannot cross the cell boundary means that the shapes are less dense near the boundary, resulting in a visible rift in the pattern at mirror lines.

It is possible to largely eliminate these rifts by a change in the code. Mirror symmetry at cell boundaries can also be achieved for circles if some of the circles are *exactly* on the boundary. We begin with a periodic boundary arrangement such as Fig. 1. We then change the code so that if any trial circle crosses the boundary it is moved perpendicular to the boundary so that it is exactly on the boundary. This new position is then tested for overlap with any previously-placed circle. The result is shown in Fig. 2b which as a 2x2 array of such cells combined to give the p2mm wallpaper group. The cell boundaries are much less obvious than in Fig. 2a. Such a structure is only possible if the shape itself is symmetric about two orthogonal axes. The code gets more complicated for a primitive cell boundary which is not a rectangle.

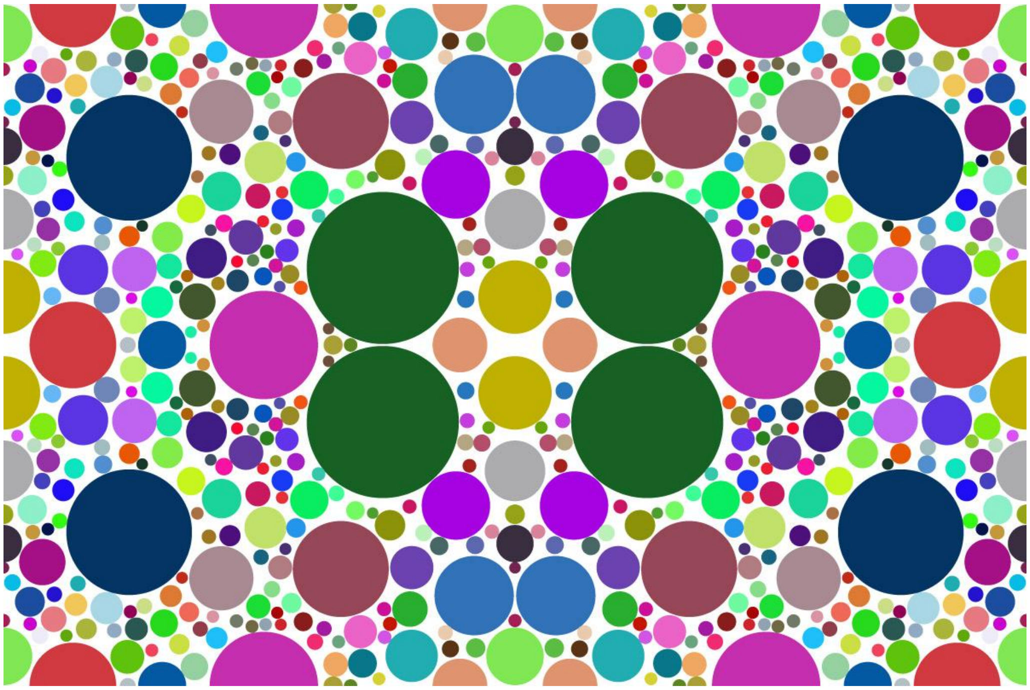


Fig. 2b. A p2mm symmetric array of circles in which the mirroring at cell boundaries has been achieved by constraining some of the circles to lie exactly on the boundary. Random color.

2. p4mm Symmetry.

This symmetry can be achieved if we place the shapes in a 45 degree triangle (the primitive unit cell) with inclusive boundaries as shown in Fig. 3. This primitive unit cell is copied with mirroring and/or rotation seven more times to produce the complete square unit cell. If the flowers were all the same color they would be mirrored across all of the vertical, horizontal, and diagonal mirror lines. It was felt to be more interesting to have four flower colors, a blue-pink set and a yellow-orange set. If the flowers are not all the same color a variety of color symmetry patterns are possible, whose number greatly expands when more colors and shapes are used. With the arrangement shown in Fig. 3 a flower mirrors to the same color across the diagonal mirror lines, while blue ↔ yellow and orange ↔ pink across the vertical mirror lines and pink ↔ yellow and orange ↔ blue across the horizontal mirror lines. If the block shown is repeated (tiled) one sees a checkerboard of blue-pink and yellow-orange regions. Gardeners sometimes try to place flowers in symmetric arrangements in a formal garden, and this can be thought of as carrying this concept to an extreme. It is an illustration of the visual impact with both order and randomness.

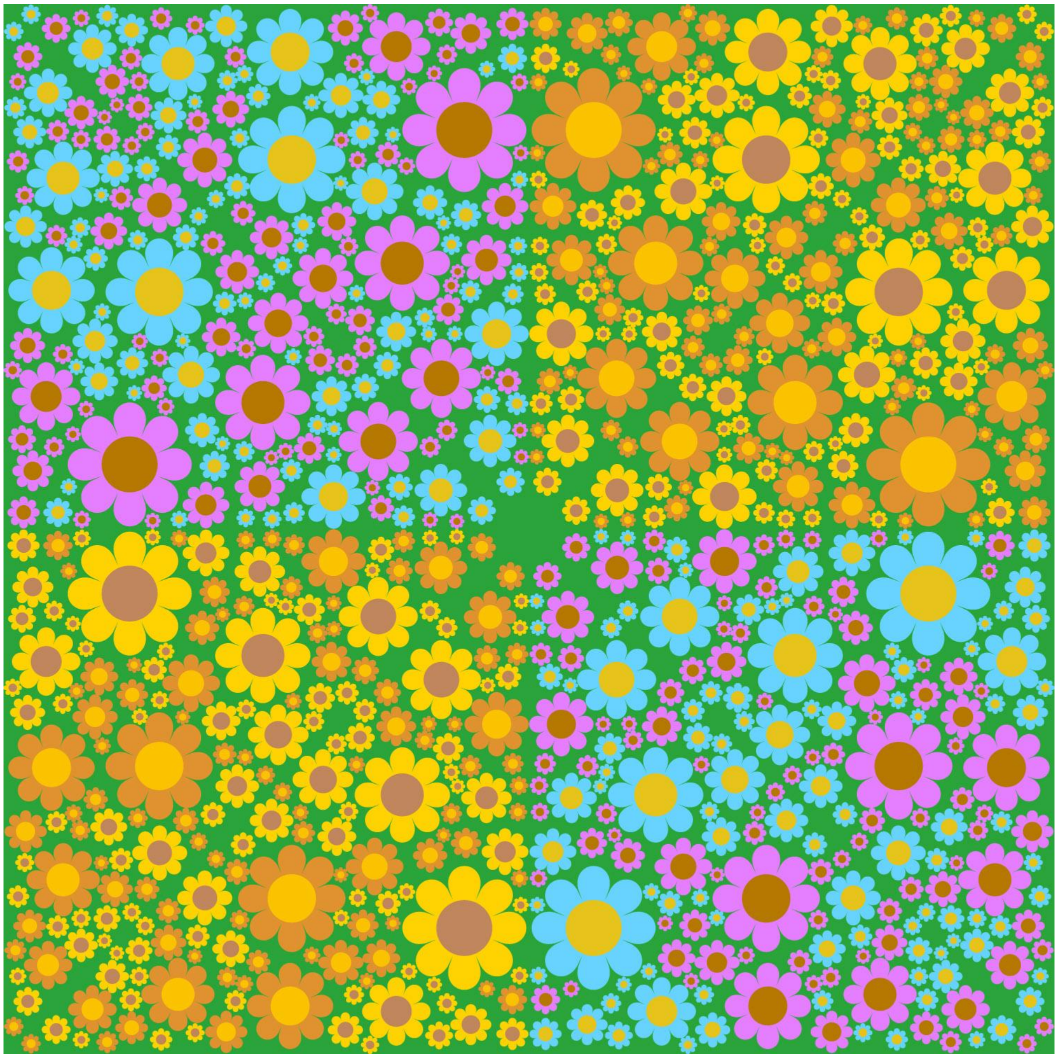


Fig. 3. A flower array with p4mm symmetry and an illustration of color symmetry.

Figure 4 shows a p4mm tiling with shapes which are 45° triangles having the same orientation as the primitive unit cell to be filled. For this reason the rifts at mirror lines are much less obvious. The small triangles have (on average) three half-size smaller triangles as near neighbors -- a pattern seen previously with equilateral triangles and quite similar to the pattern of short-range order in the Sierpinski triangles. Some Native American blankets and rungs have triangles as a motif. Others may see a similarity to cuneiform writing. The two colors are arranged so that every mirror line mirrors a given color into the same color.

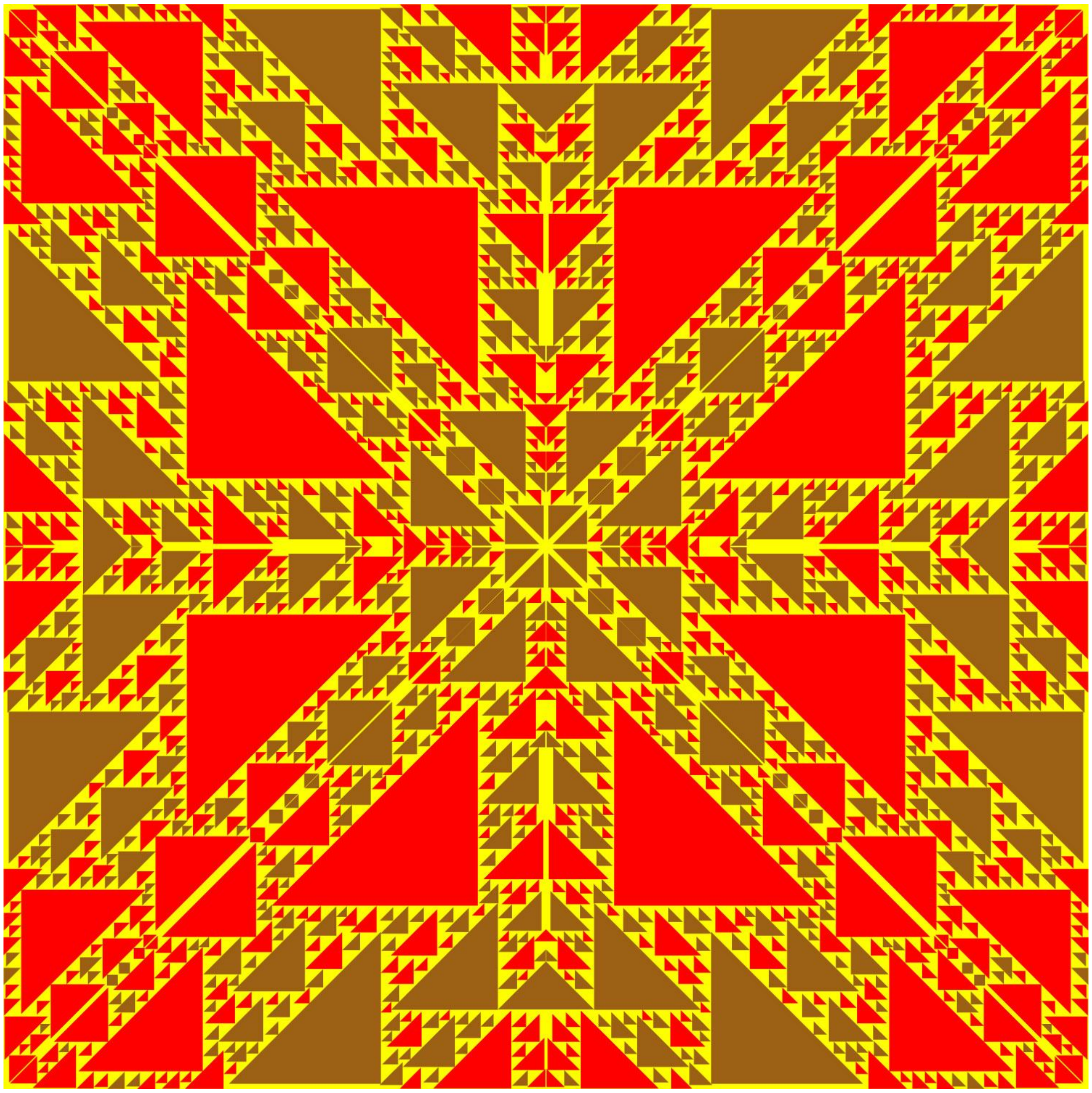
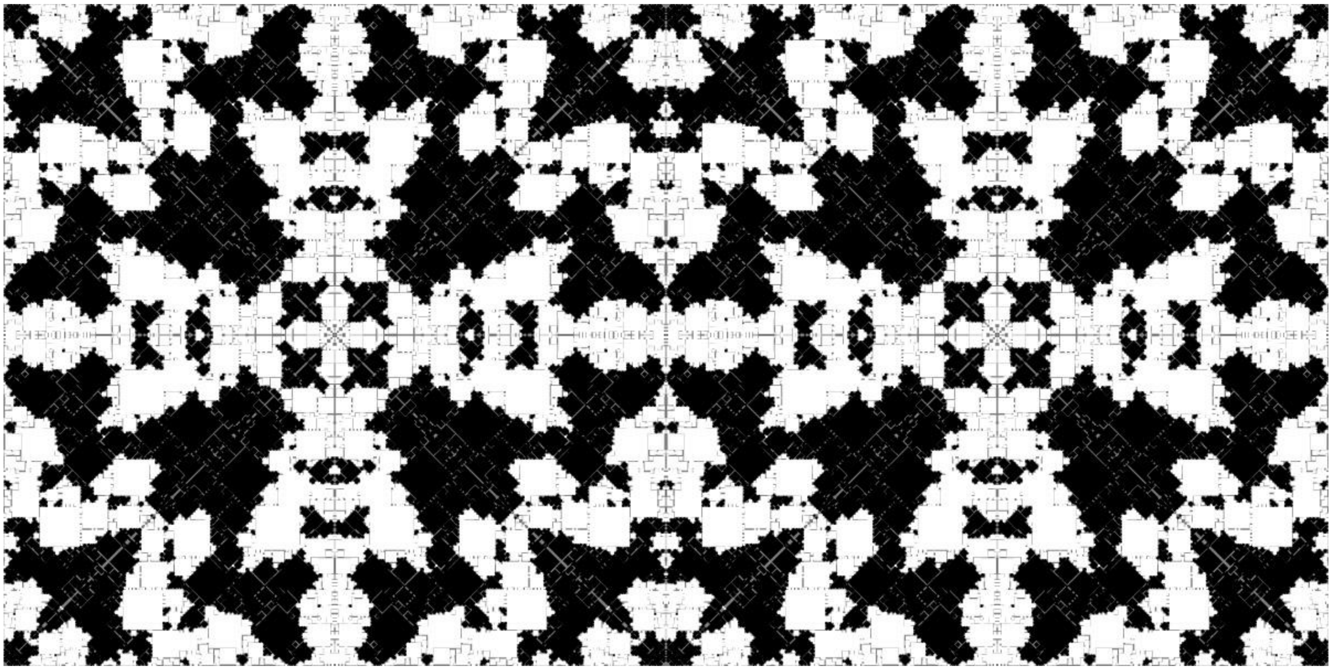
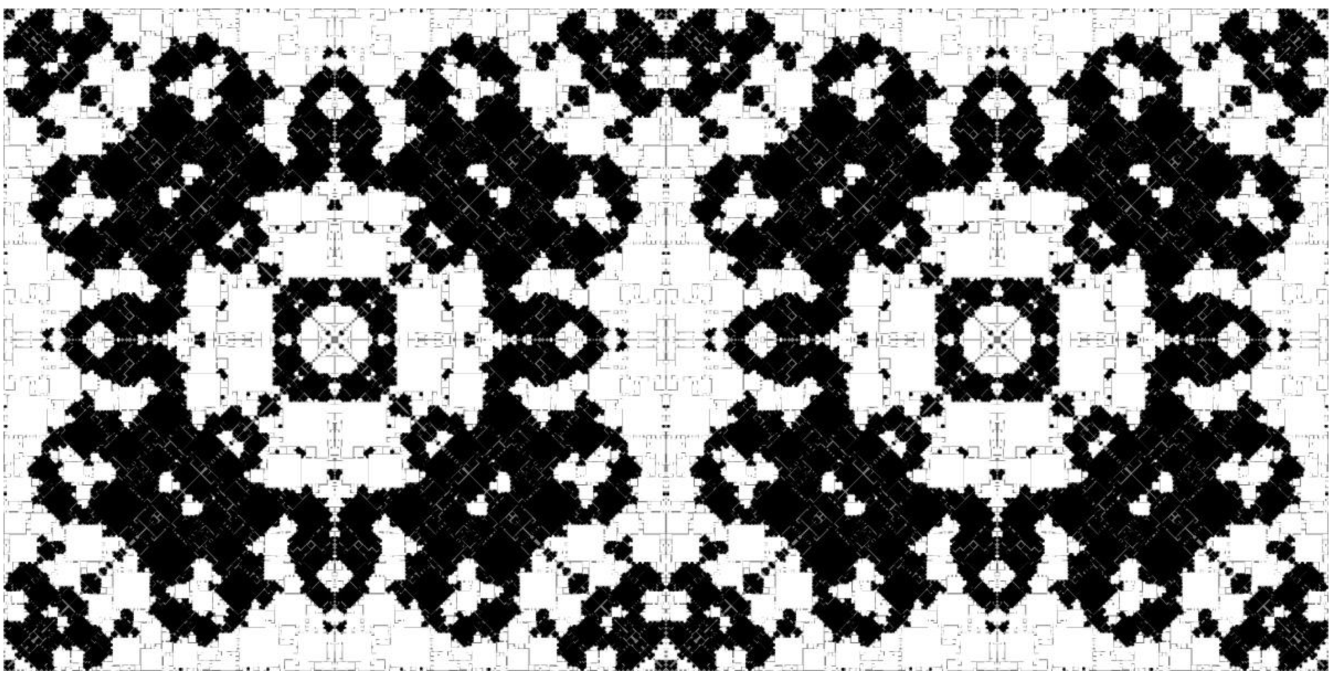


Fig. 4. A triangle array with p4mm symmetry and two colors.

One of the themes here has been the interplay of order and randomness. This is strongly visible in images made with the primitive unit cell filled with alternating squares and diamonds. Earlier studies of these patterns with rectangular boundaries showed that they divide the space quite randomly into regions of mostly squares and mostly diamonds with irregular boundaries between them. Here the squares are white and the diamonds black, with a 50% gray background. Without the imposition of symmetry such patterns just look like odd random maps, resembling the black and white patches on some dairy cows. The use of p4mm symmetry imposes bilateral symmetry at all of the mirror lines. This gives a quite different eye-brain perception of the image. Humans and most animals have bilateral symmetry and our brains are born and trained to see it. (There is a whole area of our brains which is just devoted to face recognition.) The result is that most people see "faces" and "animals" in these images, even though they are completely random except for the imposed p4mm symmetry. Some of the "faces" we recognize are black, and others white. The shape of the outline seems to be the key thing in recognition.

These images somewhat resemble the "Rohrshach" (sp?) ink blot patterns used by psychiatrists in the 1950s, although they are "intent-free" and purely random. Three examples are shown as an illustration. They tell us more about human perception than anything else. The degree to which life forms are perceived varies substantially with the parameters c and N and the percentage fill.





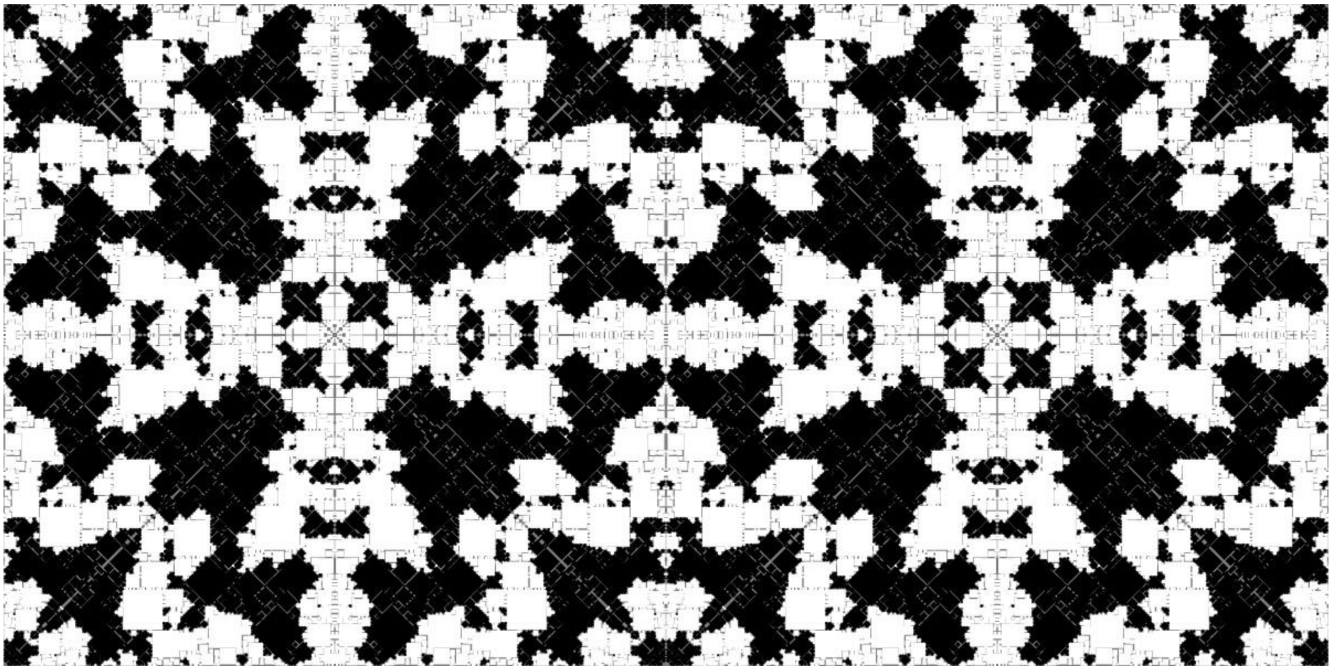


Fig. 5. Three examples of arrays of squares and diamonds with p4mm symmetry and two colors. Do you see faces or animals?

3. p6mm Symmetry.

The p6mm wallpaper group is hexagonal, and has the most symmetry elements of all of the 17 wallpaper groups. The basic scheme is shown in Fig. 6, where the primitive unit cell is a 30° triangle and the placed shapes are also 30° triangles.

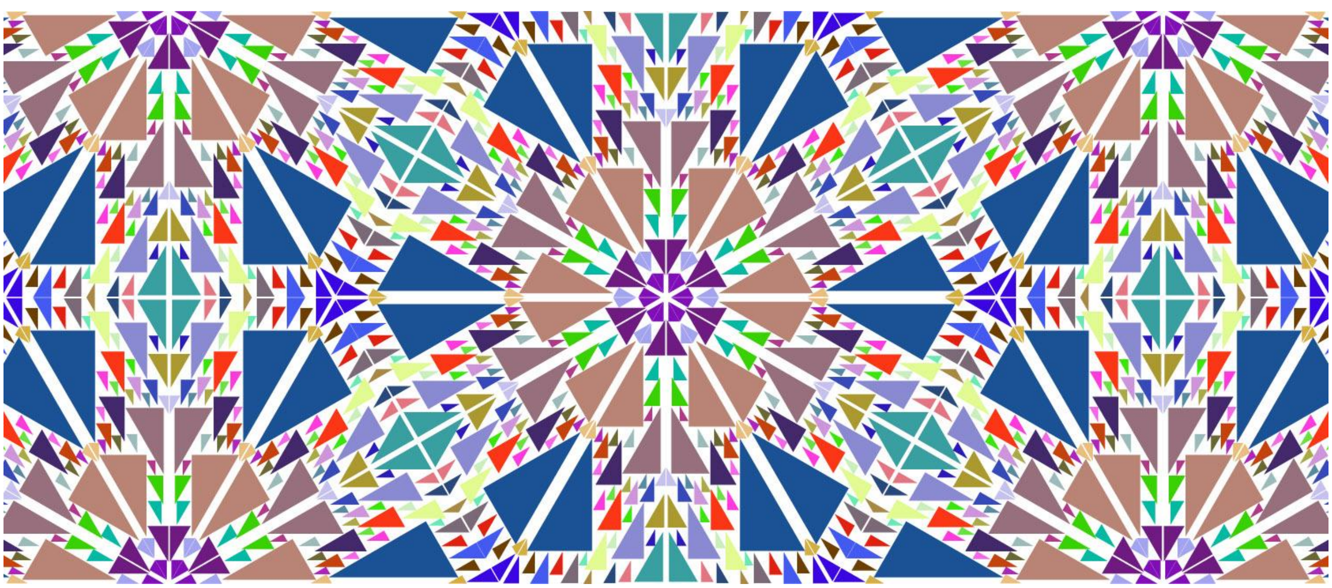


Fig. 6. A 30-degree triangle array with p6mm symmetry and random colors.

Figure 7 shows a p6mm "geometric garden" with 12-petal sunflowers against a green background. The rifts at mirror lines are clearly visible. The six mirror lines passing through the center of the pattern result in twelvefold repeats of the twelve-petal flowers so that they appear to make circles around this point. If flowers of several colors are used the number of possible color symmetry patterns increases quite rapidly.

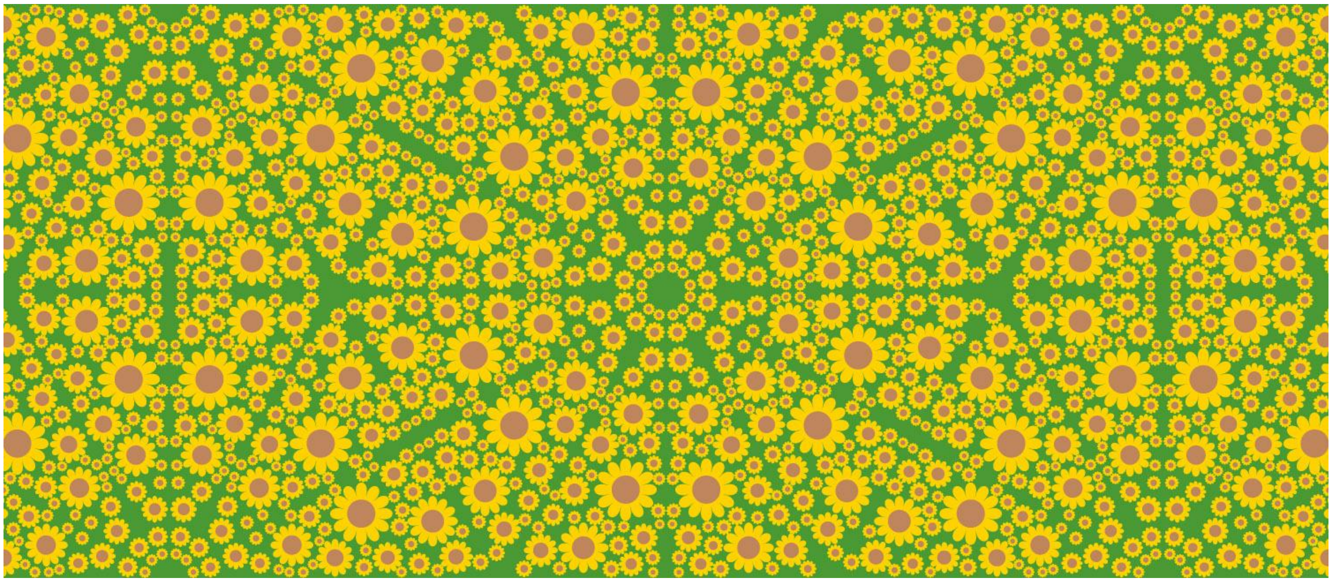


Fig. 7. A sunflower array with p6mm symmetry and identical colors.

Figure 8 shows arrows with p6mm symmetry. The arrow is made up from a square and an equilateral triangle. One of the favorable features of this arrow design is that the angle of every edge in the arrow is a multiple of 30° so that there is only a small rift at boundaries of primitive unit cells. If the arrows are thought of as signs giving directions this image illustrates the confusions of the present age.

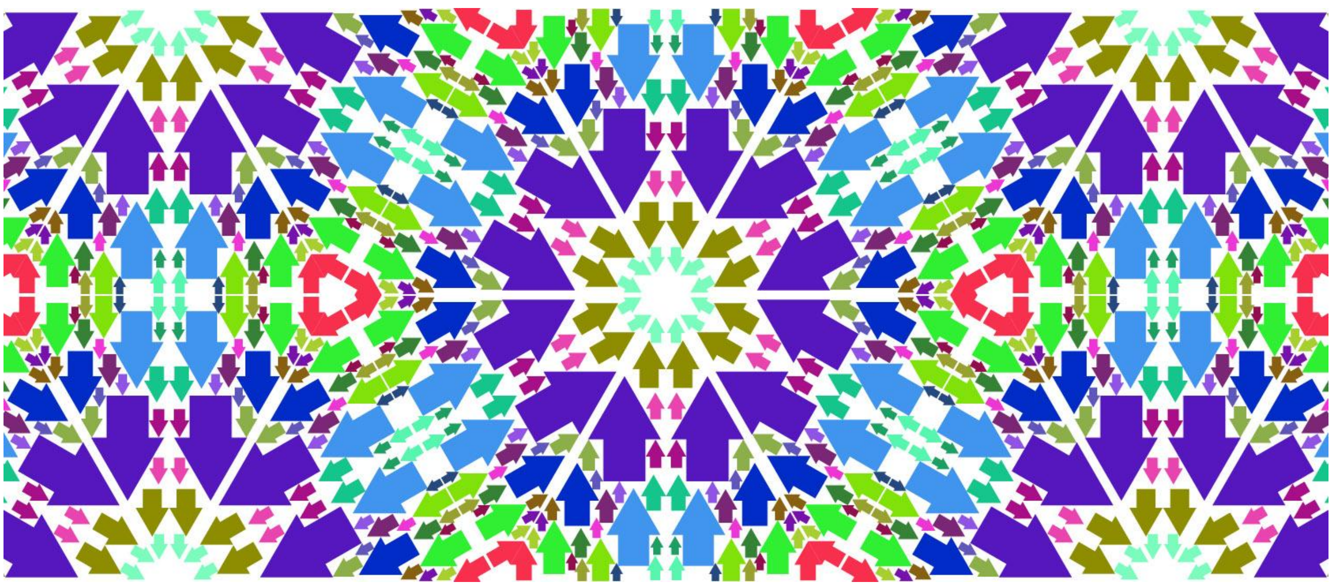


Fig. 8. An arrow array with p6mm symmetry and random colors. It could be described as a sort of periodic kaleidoscope.

Up to this point the p6mm tessellations have been constructed using shapes which lie entirely within the primitive unit cell. Consider circles. One can think of putting some of the circles precisely on the boundary (as in Fig. 2b), so that one retains the required mirror symmetry. A modified algorithm which does that was used to construct Fig. 9. The centers of the random circle locations are allowed to lie anywhere within the primitive unit cell, but if the circle crosses the boundary its center is moved perpendicular to the boundary until it is precisely on it. Circles close to a corner are moved to have their centers precisely on the corner. The test for overlap with any previous circle then proceeds as usual. A bit of thought will show that without modification such an arrangement violates the area rule since the boundary circles are shared at least 2 ways and if the usual sequence of radii is used the resulting boundary circles contribute only half the area of interior circles. The simplest way around this is to assign boundary circles a radius which is larger by a factor $\sqrt{s}$ for s-fold-shared circles than the corresponding interior-circle radius.

With this arrangement the rifts seen at mirror lines with inclusive boundaries are largely eliminated. There is an additional element of randomness here because if (say) the 5th circle is at a twelvefold sharing point it can have a radius larger than a nonshared first circle. The evidence indicates that the process continues to be nonhalting for a wide range of c and N.

The p6mm symmetry elements are not obvious here due to the blend of randomness and order. The careful observer can find twelve mirror lines passing through the center of the pattern, marked by sets of circles whose centers lie precisely along a single line.

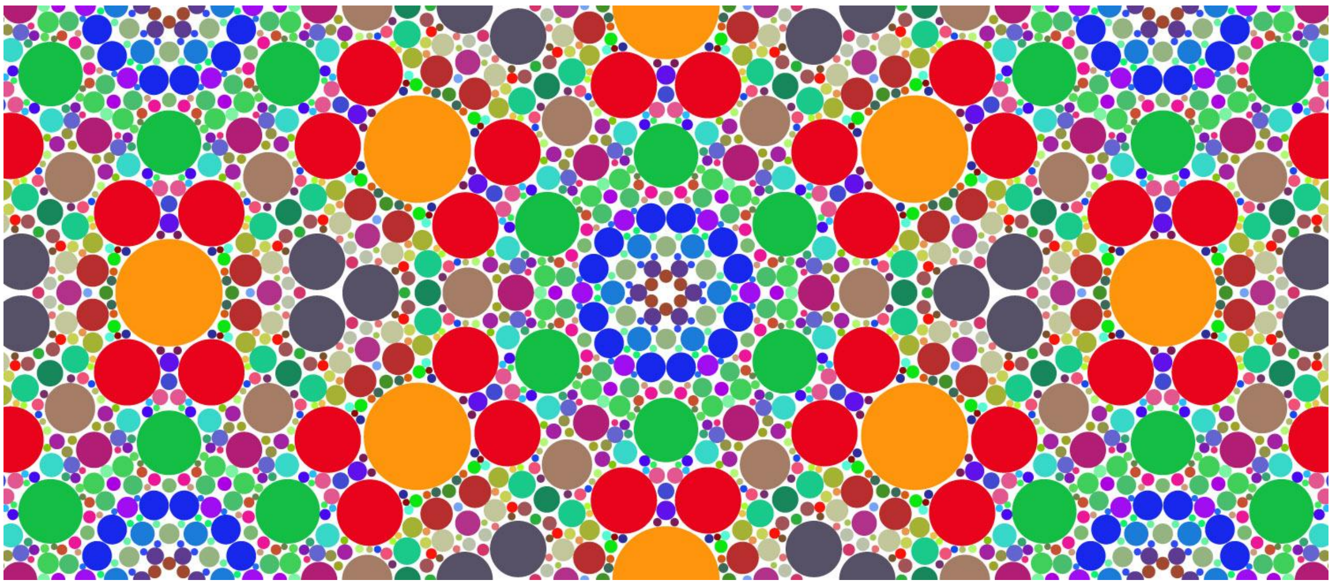


Fig. 9. A p6mm circle array in which "boundary" circles are placed precisely on the mirror lines.

Figure 10 also shows p6mm circles but with the "grouted look". The author has long noted that when c falls toward 1 there is much empty space and only a modest number of trials are needed to place a shape. The thought came "Can one impose an additional constraint of a finite minimum separation under such conditions?" Consider the placement of circle i with radius r_i. One can insist that the random trials must find a place with spacing βr_i from any previously placed circle, but put the radius r_i in the data base. This has the effect of spacing the circles farther apart, i.e., what a tile craftsman would call the "grouted look". When log-log plots are made of cumulative trials versus placed shape number it is seen that the trend supports nonhalting. Since the usual area rule is followed such a construction remains space-filling in the limit, even though none of the circles comes close to any other. When β is greater than 1, the largest usable c value falls.

Visually Fig. 10 gives a rather different impression than images generated by the simple nonoverlap rule such as Fig. 9. They eye picks up each circle individually. In Figure 10 the p6mm symmetry isn't the first thing that the eye perceives. The straddling circles on the mirror lines serve to conceal them.

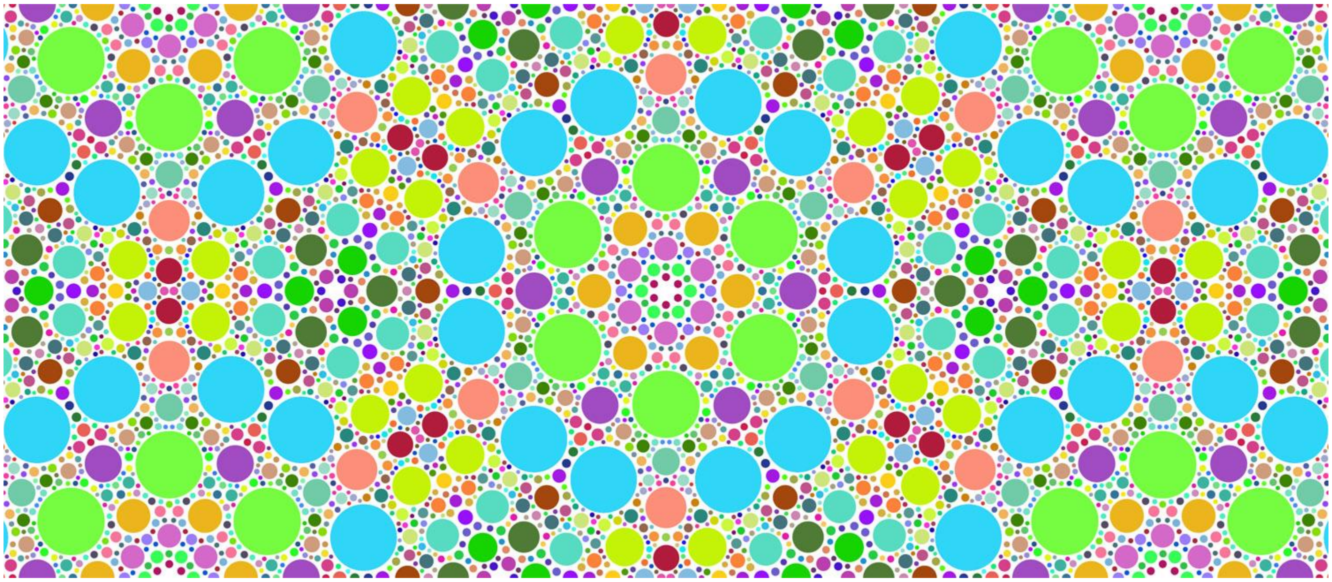


Fig. 10. A p6mm circle array in which "boundary" circles are placed precisely on the mirror lines and the circles are "mutually avoiding" circles with a finite minimum spacing from each other.

4. Comments.

The object of this work was to study the combination of a random statistical geometry pattern and a highly symmetric tessellation -- randomness blending with order. The shapes used are mostly rather simple, but they illustrate the possibilities. A wide variety of shapes should be usable in this way with enough patience in coding.

It was anticipated that the order-and-disorder combination in these images would provide an engaging image, and this is judged to be true. With a modest amount of coding the principles illustrated here should be extendable to all 17 wallpaper groups.

The most common scientific use of space groups is in crystallography, where the unit cell rarely contains more than 30-40 elements (atoms). The patterns shown here can greatly exceed that number of elements and thus present a different appearance. Art tessellations[1] of quite inventive shapes have been created, but they usually contain only a single element -- most commonly a somewhat distorted person or animal. The tessellations shown here are rather different from either of those examples.

5. References

[1] J. Shier and P. Bourke, Computer Graphics Forum, **32**, Issue 8, 89 (2013). This is the first publication in a refereed journal.

[2] Wikipedia has an informative page on wallpaper groups for those not familiar with the concept. This system provides precise names (such as p4mm and p6mm) for all of the 17 possible symmetric spatial arrangements (space groups) in two dimensions.

[3] D. Dunham and J. Shier, Proceedings of the 2014 Bridges Conference; Seoul, Korea, p. 79.

[4] J. Shier, Hyperseeing (Proceedings of the 2011 ISAMA Conference), June, 131 (2011).

[1] If we consider decorative tessellations used for fabric and wallpaper the patterns, they often contain a quite large number of objects in the unit cell. Such patterns almost always belong to the simplest wallpaper group with translational symmetry only (as in Fig. 1).

[5] P. Bourke, web site http://paulbourke.net/fractals/

[6] J. Shier, web site http://john-art.com/